Numerical Solution of Fuzzy Differential Equations
by Runge Kutta Method of Order Five

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Abstract

In this paper numerical algorithms for solving ‘fuzzy ordinary differential equations’ based on Seikkala derivative of fuzzy process [9], are considered. A numerical method based on the Runge-Kutta method of order five in detail is discussed and this is followed by a complete error analysis. The algorithm is illustrated by solving some linear and nonlinear Fuzzy Cauchy Problems.

Keywords: Fuzzy Differential Equation, Runge-Kutta Method of Order Five, Fuzzy Cauchy Problem

1 Introduction

The concept of fuzzy derivative was first introduced by S.L. Chang, L.A. Zadeh in [3] . It was followed up by D. Dubois, H. Prade in [4] , who defined and used the extension principle. The fuzzy differential equation and the initial value problem were regularly treated by O. Kelva in [7, 8] and by S. Seikkala in [9]. The numerical method for solving fuzzy differential equations is introduced by M.Ma, M.Friedman, A. Kandel in [12] by the standard Euler Method and by authors in [1, 2] by Taylor method.

The structure of this Chapter organizes as follows. In section 2 some basic results on fuzzy numbers and definition of a fuzzy derivative, which have been discussed by S. Seikkala in [9], are given. In section 3 we define the problem, this is a fuzzy Cauchy problem whose numerical solution is the main interest of this chapter. The numerically solving fuzzy differential equation by the Runge-Kutta method of order 5 is discussed in section 4. The proposed
algorithm is illustrated by solving some examples in section 5 and conclusion is in section 6.

2 Preliminary Notes

Consider the initial value problem

\[
\begin{align*}
    y'(t) &= f(t, y(t)); \quad a \leq t \leq b, \\
    y(a) &= \alpha,
\end{align*}
\]  

(1)

The basis of all Runge-Kutta method is to express the difference between the value of \( y \) at \( t_{n+1} \) and \( t_n \) as

\[
    y_{n+1} - y_n = \sum_{i=1}^{m} w_i k_i,
\]  

(2)

where for \( i = 1, 2, \ldots, m \), the \( w_i's \) are constants and

\[
    k_i = h \cdot f(t_n + \alpha_i h, y_n + \sum_{j=1}^{i-1} \beta_{ij} k_j).
\]  

(3)

Equation (2) is to be exact for powers of \( h \) through \( h^m \), because it is to be coincident with Taylor series of order \( m \). Therefore, the truncation error \( T_m \), can be written as

\[
    T_m = \gamma_m h^{m+1} + O(h^{m+2}).
\]

The true magnitude of \( \gamma_m \) will generally be much less than the bound of theorem 2.1 Thus, if the \( O(h^{m+2}) \) term is small compared with \( \gamma_m h^{m+1} \), as we expect, to be so if \( h \) is small, then the bound on \( \gamma_m h^{m+1} \), will usually be a bound on the error as a whole. The famous nonzero constants \( \alpha_i, \beta_{ij} \) in the Runge Kutta method of order 5 are

\[
    \begin{align*}
        \alpha_1 &= 0, \quad \alpha_2 = \alpha_3 = \frac{1}{3}, \quad \alpha_4 = \frac{1}{2}, \quad \alpha_5 = 1, \quad \beta_{21} = \frac{1}{3}, \quad \beta_{31} = \beta_{32} = \frac{1}{6}, \quad \beta_{41} = \frac{1}{8}, \\
        \beta_{42} &= 0, \quad \beta_{43} = \frac{3}{8}, \quad \beta_{51} = \frac{1}{2}, \quad \beta_{52} = 0, \quad \beta_{53} = \frac{3}{2}, \quad \beta_{54} = 2.
    \end{align*}
\]
where \( m = 5 \). Hence we have,

\[
y_0 = \alpha,
\]

\[
k_1 = h \cdot f(t_i, y_i),
\]

\[
k_2 = h \cdot f(t_i + \frac{h}{3}, y_i + \frac{k_1}{3}),
\]

\[
k_3 = h \cdot f(t_i + \frac{h}{6}, y_i + \frac{k_1}{6} + \frac{k_2}{6}),
\]

\[
k_4 = h \cdot f(t_i + \frac{h}{2}, y_i + \frac{k_1}{2} + \frac{3k_2}{8}),
\]

\[
k_5 = h \cdot f(t_i + h, y_i + \frac{k_1}{2} - \frac{3k_2}{2} + 2k_3),
\]

\[
y_{i+1} = y_i + \frac{1}{6}(k_1 + 4k_4 + k_5),
\]

where

\[
a = t_0 \leq t_1 \leq \ldots \leq t_N = b \text{ and } h = \frac{(b-a)}{N} = t_{i+1} - t_i. \tag{5}
\]

**Theorem 2.1** Let \( f(t, y) \) belong to \( C^4[a, b] \) and let its partial derivatives are bounded and assume there exists, \( L, M, \) positive numbers, such that

\[
|f(t, y)| < M, \quad \left| \frac{\partial^{i+j} f}{\partial t^i \partial y^j} \right| < \frac{L^{i+j}}{M^{j-1}}, \quad i+j \leq m,
\]

then in the Runge-Kutta method of order 5, \( y(t_{i+1}) - y_{i+1} \approx \frac{11987}{12960} h^6 M L^5 + O(h^7) \)

A triangular fuzzy number \( v \), is defined by three numbers \( a_1 < a_2 < a_3 \) where the graph of \( v(x) \), the membership function of the fuzzy number \( v \), is a triangle with base on the interval \([a_1, a_2]\) and vertex at \( x = a_2 \). We specify \( v \) as \( (a_1/a_2/a_3) \). We will write: (1)\( v > 0 \) if \( a_1 > 0 \); (2)\( v \geq 0 \) if \( a_1 \geq 0 \); (3)\( v < 0 \) if \( a_3 < 0 \); and (4)\( v \leq 0 \) if \( a_3 \leq 0 \).

Let \( E \) be the set of all upper semicontinuous normal convex fuzzy numbers with bounded \( r \)-level intervals. It means that if \( v \in E \) then the \( r \)-level set

\[
[v]_r = \{ s \mid v(s) \geq r \}, \quad 0 < r \leq 1,
\]

is a closed bounded interval which is denoted by

\[
[v]_r = [v_1(r), v_2(r)].
\]
Let \( I \) be a real interval. A mapping \( x: I \rightarrow E \) is called a fuzzy process and its \( r \)-level set is denoted by
\[
[x(t)]_r = [x_1(t; r), x_2(t; r)], \quad t \in I, \quad r \in (0, 1].
\]
The derivative \( x'(t) \) of a fuzzy process \( x(t) \) is defined by
\[
[x'(t)]_r = [x'_1(t; r), x'_2(t; r)], \quad t \in I, \quad r \in (0, 1],
\]
provided that this equation defines a fuzzy number, as in Seikkala [9].

Lemma 2.2 Let \( v, w \in E \) and \( s \) scalar, then for \( r \in (0, 1] \)
\[
[v + w]_r = [v_1(r) + w_1(r), v_2(r) + w_2(r)],
\]
\[
[v - w]_r = [v_1(r) - w_1(r), v_2(r) - w_2(r)],
\]
\[
[v \cdot w]_r = \left[ \min \{v_1(r) \cdot w_1(r), v_1(r) \cdot w_2(r), v_2(r) \cdot w_1(r), v_2(r) \cdot w_2(r) \} \right],
\]
\[
[s \cdot v]_r = s [v]_r.
\]

3 A Fuzzy Cauchy Problem

Consider the fuzzy initial value problem
\[
\begin{cases}
    y'(t) = f(t, y(t)), & t \in I = [0, T], \\
    y(0) = y_0,
\end{cases}
\]
where \( f \) is a continuous mapping from \( R_+ \times R \) into \( R \) and \( y_0 \in E \) with \( r \)-level sets
\[
[y_0]_r = [y_1(0; r), y_2(0; r)], \quad r \in (0, 1].
\]
The extension principle of Zadeh leads to the following definition of \( f(t, y) \) when \( y = y(t) \) is a fuzzy number
\[
f(t, y)(s) = \sup \{y(\tau) \mid s = f(t, \tau)\}, \quad s \in R.
\]
It follows that
\[
[f(t, y)]_r = [f_1(t, y; r), f_2(t, y; r)], \quad r \in (0, 1],
\]
where
\[
f_1(t, y; r) = \min \{f(t, u) \mid u \in [y_1(r), y_2(r)]\},
\]
\[
f_2(t, y; r) = \max \{f(t, u) \mid u \in [y_1(r), y_2(r)]\}.
\]
**Theorem 3.1** Let $f$ satisfy

$$|f(t, v) - f(t, \bar{v})| \leq g(t, |v - \bar{v}|), \quad t \geq 0, \quad v, \bar{v} \in R,$$

where $g : R_+ \times R_+ \rightarrow R$ is a continuous mapping such that $r \rightarrow g(t, r)$ is nondecreasing, the initial value problem

$$u'(t) = g(t, u(t)), \quad u(0) = u_0,$$

has a solution on $R_+$ for $u_0 > 0$ and that $u(t) = 0$ is the only solution of (8) for $u_0 = 0$. Then the fuzzy initial value problem (6) has a unique fuzzy solution.

4 The Runge-Kutta Method of Order Five

Let the exact solution $[Y(t)]_r = [Y_1(t; r), Y_2(t; r)]$ is approximated by some $[y(t)]_r = [y_1(t; r), y_2(t; r)]$. From (2),(3) we define

$$y_1(t_{n+1}; r) - y_1(t_{n}; r) = \sum_{i=1}^{5} w_i k_{i,1}(t_{n}, y(t_{n}; r)),$$

$$y_2(t_{n+1}; r) - y_2(t_{n}; r) = \sum_{i=1}^{5} w_i k_{i,2}(t_{n}, y(t_{n}; r)).$$

where the $w_i$’s are constants and

$$[k_i(t, y(t; r))]_r = [k_{i,1}(t, y(t, r)), k_{i,2}(t, y(t, r))], \quad i = 1, 2, 3, 4, 5$$

$$k_{i,1}(t, y(t, r)) = h.f(t_n + \alpha_i h, y_1(t_n) + \sum_{j=1}^{i-1} \beta_{ij} k_{j,1}(t_n, y(t_n; r))),$$

$$k_{i,2}(t, y(t, r)) = h.f(t_n + \alpha_i h, y_2(t_n) + \sum_{j=1}^{i-1} \beta_{ij} k_{j,2}(t_n, y(t_n; r))).$$
and
\[ k_{1,1}(t, y(t; r)) = \min \{h.f(t, u) | u \in [y_1(t; r), y_2(t; r)]\}, \]
\[ k_{1,2}(t, y(t; r)) = \max \{h.f(t, u) | u \in [y_1(t; r), y_2(t; r)]\}, \]
\[ k_{2,1}(t, y(t; r)) = \min \{h.f(t + \frac{k}{3}, u) | u \in [z_{1,1}(t, y(t; r)), z_{1,2}(t, y(t; r))]\}, \]
\[ k_{2,2}(t, y(t; r)) = \max \{h.f(t + \frac{k}{3}, u) | u \in [z_{1,1}(t, y(t; r)), z_{1,2}(t, y(t; r))]\}, \]
\[ k_{3,1}(t, y(t; r)) = \min \{h.f(t + \frac{k}{3}, u) | u \in [z_{2,1}(t, y(t; r)), z_{2,2}(t, y(t; r))]\}, \]
\[ k_{3,2}(t, y(t; r)) = \max \{h.f(t + \frac{k}{3}, u) | u \in [z_{2,1}(t, y(t; r)), z_{2,2}(t, y(t; r))]\}, \]
\[ k_{4,1}(t, y(t; r)) = \min \{h.f(t + \frac{k}{3}, u) | u \in [z_{3,1}(t, y(t; r)), z_{3,2}(t, y(t; r))]\}, \]
\[ k_{4,2}(t, y(t; r)) = \max \{h.f(t + \frac{k}{3}, u) | u \in [z_{3,1}(t, y(t; r)), z_{3,2}(t, y(t; r))]\}, \]
\[ k_{5,1}(t, y(t; r)) = \min \{h.f(t + h, u) | u \in [z_{4,1}(t, y(t; r)), z_{4,2}(t, y(t; r))]\}, \]
\[ k_{5,2}(t, y(t; r)) = \max \{h.f(t + h, u) | u \in [z_{4,1}(t, y(t; r)), z_{4,2}(t, y(t; r))]\}. \]

where in the Runge-Kutta method of order 5,
\[ z_{1,1}(t, y(t; r)) = y_1(t; r) + \frac{1}{3}k_{1,1}(t, y(t; r)), \]
\[ z_{1,2}(t, y(t; r)) = y_2(t; r) + \frac{1}{3}k_{1,2}(t, y(t; r)), \]
\[ z_{2,1}(t, y(t; r)) = y_1(t; r) + \frac{1}{6}k_{1,1}(t, y(t; r)) + \frac{1}{6}k_{2,1}(t, y(t; r)), \]
\[ z_{2,2}(t, y(t; r)) = y_2(t; r) + \frac{1}{6}k_{1,2}(t, y(t; r)) + \frac{1}{6}k_{2,2}(t, y(t; r)), \]
\[ z_{3,1}(t, y(t; r)) = y_1(t; r) + \frac{1}{8}k_{1,1}(t, y(t; r)) + \frac{3}{8}k_{3,1}(t, y(t; r)), \]
\[ z_{3,2}(t, y(t; r)) = y_2(t; r) + \frac{1}{8}k_{1,2}(t, y(t; r)) + \frac{3}{8}k_{3,2}(t, y(t; r)), \]
\[ z_{4,1}(t, y(t; r)) = y_1(t; r) + \frac{1}{8}k_{1,1}(t, y(t; r)) - \frac{3}{32}k_{3,1}(t, y(t; r)) + 2k_{4,1}(t, y(t; r)), \]
\[ z_{4,2}(t, y(t; r)) = y_2(t; r) + \frac{1}{8}k_{1,2}(t, y(t; r)) - \frac{3}{32}k_{3,2}(t, y(t; r)) + 2k_{4,2}(t, y(t; r)). \]
Define,
\[
F[t, y(t; r)] = k_{1,1}(t, y(t; r)) + 4k_{4,1}(t, y(t; r)) + k_{5,1}(t, y(t; r))
\]
\[
G[t, y(t; r)] = k_{1,2}(t, y(t; r)) + 4k_{4,2}(t, y(t; r)) + k_{5,2}(t, y(t; r)).
\] (13)

The exact and approximate solutions at \( t = t_n, 0 \leq n \leq N \) are denoted by \[ Y(t_n) = [Y_1(t_n; r), Y_2(t_n; r)] \] and \[ y(t_n) = [y_1(t_n; r), y_2(t_n; r)] \], respectively. The solution is calculated by grid points at (5). By (9), (13) we have
\[
Y_1(t_{n+1}; r) \approx Y_1(t_n; r) + \frac{1}{6} F[t_n, Y(t_n; r)], \\
Y_2(t_{n+1}; r) \approx Y_2(t_n; r) + \frac{1}{6} G[t_n, Y(t_n; r)].
\] (14)

We define
\[
y_1(t_{n+1}; r) = y_1(t_n; r) + \frac{1}{6} F[t_n, y(t_n; r)], \\
y_2(t_{n+1}; r) = y_2(t_n; r) + \frac{1}{6} G[t_n, y(t_n; r)].
\] (15)

The following lemmas will be applied to show convergence of these approximates i.e.,
\[
\lim_{h \to 0} y_1(t; r) = Y_1(t; r), \\
\lim_{h \to 0} y_2(t; r) = Y_2(t; r).
\]

**Lemma 4.1** Let the sequence of numbers \( \{W_n\}_{n=0}^N \) satisfy
\[
|W_{n+1}| \leq A|W_n| + B, \quad 0 \leq n \leq N - 1,
\]
for some given positive constants \( A \) and \( B \). Then
\[
|W_n| \leq A^n|W_0| + B\frac{A^n - 1}{A - 1}, \quad 0 \leq n \leq N.
\]

**Lemma 4.2** Let the sequence of numbers \( \{W_n\}_{n=0}^N, \{V_n\}_{n=0}^N \) satisfy
\[
|W_{n+1}| \leq |W_n| + A \max\{|W_n|, |V_n|\} + B, \\
|V_{n+1}| \leq |V_n| + A \max\{|W_n|, |V_n|\} + B.
\]
for some given positive constants \( A \) and \( B \), and denote
\[
U_n = |W_n| + |V_n|, \quad 0 \leq n \leq N.
\]
Then
\[ |U_n| \leq \frac{\overline{A}^n U_0 + B(\overline{A})^n - 1}{A - 1}, \quad 0 \leq n \leq N, \]

where \( \overline{A} = 1 + 2A \) and \( B = 2B. \)

Let \( F(t, u, v) \) and \( G(t, u, v) \) are obtained by substituting \( [y(t)]_r = [u, v] \) in (13),

\[
F(t, u, v) = k_{1,1}(t, u, v) + 4k_{4,1}(t, u, v) + k_{5,1}(t, u, v) \\
G(t, u, v) = k_{1,2}(t, u, v) + 4k_{4,2}(t, u, v) + k_{5,2}(t, u, v).
\]

The domain where \( F \) and \( G \) are defined is therefore
\[ K = \{(t, u, v)| 0 \leq t \leq T, \quad -\infty < v < \infty, \quad -\infty < u < v\}. \]

**Theorem 4.3** Let \( F(t, u, v) \) and \( G(t, u, v) \) belong to \( C^4(K) \) and let the partial derivatives of \( F \) and \( G \) be bounded over \( K \). Then, for arbitrary fixed \( r, 0 \leq r \leq 1 \), the approximately solutions (14) converge to the exact solutions \( Y_1(t; r) \) and \( Y_2(t; r) \) uniformly in \( t \).

**Proof:**

It is sufficient to show
\[
\lim_{h \to 0} y_1(t_n; r) = Y_1(t_n; r), \\
\lim_{h \to 0} y_2(t_n; r) = Y_2(t_n; r),
\]

where \( t_N = T. \) For \( n = 0, 1, \ldots, N - 1 \), by using Taylor theorem we get
\[
Y_1(t_{n+1}; r) = Y_1(t_n; r) + \frac{1}{6}F[t_n, Y_1(t_n; r), Y_2(t_n; r)] + \frac{11987}{12960}h^6ML^5 + O(h^7),
\]

\[
Y_2(t_{n+1}; r) = Y_2(t_n; r) + \frac{1}{6}G[t_n, Y_1(t_n; r), Y_2(t_n; r)] + \frac{11987}{12960}h^6ML^5 + O(h^7),
\]

denote
\[
W_n = Y_1(t_n; r) - y_1(t_n; r), \\
V_n = Y_2(t_n; r) - y_2(t_n; r).
\]

Hence from (15) and (16)
\[
W_{n+1} = W_n + \frac{1}{6}\left\{ F[t_n, Y_1(t_n; r), Y_2(t_n; r)] - F[t_n, y_1(t_n; r), y_2(t_n; r)] \right\} \\
\quad + \frac{11987}{12960}h^6ML^5 + O(h^7),
\]

\[
V_{n+1} = V_n + \frac{1}{6}\left\{ G[t_n, Y_1(t_n; r), Y_2(t_n; r)] - G[t_n, y_1(t_n; r), y_2(t_n; r)] \right\} \\
\quad + \frac{11987}{12960}h^6ML^5 + O(h^7).
\]
Then
\[ |W_{n+1}| \leq |W_n| + \frac{1}{3} L \max \{|W_n|, |V_n|\} + \frac{11987}{12960} h^6 M L^5 + O(h^7), \]

\[ |V_{n+1}| \leq |V_n| + \frac{1}{3} L \max \{|W_n|, |V_n|\} + \frac{11987}{12960} h^6 M L^5 + O(h^7), \]

for \( t \in [0, T] \) and \( L > 0 \) is a bound for the partial derivatives of \( F \) and \( G \). Thus by lemma 4.2

\[ |W_n| \leq (1 + \frac{2}{3} L h)^n |U_0| + \left( \frac{11987}{6480} h^6 M L^5 + O(h^7) \right) \frac{(1 + \frac{2}{3} L h)^n - 1}{\frac{2}{3} L h}, \]

\[ |V_n| \leq (1 + \frac{2}{3} L h)^n |U_0| + \left( \frac{11987}{6480} h^6 M L^5 + O(h^7) \right) \frac{(1 + \frac{2}{3} L h)^n - 1}{\frac{2}{3} L h}, \]

where \( |U_0| = |W_0| + |V_0| \). In particular

\[ |W_n| \leq (1 + \frac{2}{3} L h)^n |U_0| + \left( \frac{11987}{4320} h^5 M L^5 + O(h^6) \right) \frac{(1 + \frac{2}{3} L h)^n - 1}{L}, \]

\[ |V_n| \leq (1 + \frac{2}{3} L h)^n |U_0| + \left( \frac{11987}{4320} h^5 M L^5 + O(h^6) \right) \frac{(1 + \frac{2}{3} L h)^n - 1}{L}. \]

Since \( W_0 = V_0 = 0 \), we obtain

\[ |W_n| \leq \left( \frac{11987}{4320} M L^4 \right) \frac{e^{\frac{2}{3} L T} - 1}{L} h^4 + O(h^6), \]

\[ |V_n| \leq \left( \frac{11987}{4320} M L^4 \right) \frac{e^{\frac{2}{3} L T} - 1}{L} h^4 + O(h^6), \]

and if \( h \to 0 \) we get \( W_N \to 0 \) and \( V_N \to 0 \) which completes the proof.
5 NUMERICAL EXAMPLES

Example 5.1 Consider the fuzzy initial value problem, [12],

\[
\begin{align*}
    y'(t) &= y(t), \quad t \in I = [0, 1], \\
    y(0) &= (0.75 + 0.25r, 1.125 - 0.125r), \quad 0 < r \leq 1.
\end{align*}
\]

By using the Runge-Kutta method of order 5, we have

\[
\begin{align*}
    y_1(t_{n+1}; r) &= y_1(t_n; r)\left[1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24} + \frac{h^5}{144}\right], \\
    y_2(t_{n+1}; r) &= y_2(t_n; r)\left[1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24} + \frac{h^5}{144}\right].
\end{align*}
\]

The exact solution is given by

\[
Y_1(t; r) = y_1(0; r)e^t, \quad Y_2(t; r) = y_2(0; r)e^t,
\]

which at \( t = 1 \),

\[
Y_1(1; r) = (0.75 + 0.25r)e, \quad (1.125 - 0.125r)e, \quad 0 < r \leq 1.
\]

The exact and approximate solutions by Improved Euler method and the Runge Kutta method of order 5, are compared and plotted at \( t = 1 \) in figure 1.

**Table 1**

<table>
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<th>( r )</th>
<th>Improved Euler’s Method</th>
<th>Runge Kutta Method of order 5</th>
<th>Exact Solution</th>
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<td>( y_2(t; r) )</td>
<td>( y_1(t; r) )</td>
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</table>
Example 5.2 Consider the fuzzy initial value problem

\[ y'(t) = c_1 y^2(t) + c_2, \quad y(0) = 0, \]

where \( c_i > 0 \), for \( i = 1, 2 \) are triangular fuzzy numbers, [13].

The exact solution is given by

\[
Y_1(t; r) = l_1(r)\tan(w_1(r)t), \\
Y_2(t; r) = l_2(r)\tan(w_2(r)t),
\]

with

\[
l_1(r) = \sqrt{c_{2,1}(r)/c_{1,1}(r)}, \quad l_2(r) = \sqrt{c_{2,2}(r)/c_{1,2}(r)}, \\
w_1(r) = \sqrt{c_{1,1}(r)c_{2,1}(r)}, \quad w_2(r) = \sqrt{c_{1,2}(r)c_{2,2}(r)},
\]

where

\[
[c_1]_r = [c_{1,1}(r), c_{1,2}(r)] \quad \text{and} \quad [c_2]_r = [c_{2,1}(r), c_{2,2}(r)]
\]

\[
c_{1,1}(r) = 0.5 + 0.5r, \quad c_{1,2}(r) = 1.5 - 0.5r, \\
c_{2,1}(r) = 0.75 + 0.25r, \quad c_{2,2}(r) = 1.25 - 0.25r.
\]
The $r-$level sets of $y'(t)$ are
\[ Y'_1(t; r) = c_{2,1}(r)\sec^2(w_1(r)t), \]
\[ Y'_2(t; r) = c_{2,2}(r)\sec^2(w_2(r)t), \]
which defines a fuzzy number. We have
\[ f_1(t, y; r) = \min\{c_1,u^2 + c_2|u \in [y_1(t; r), y_2(t; r)], c_1 \in [c_{1,1}(r), c_{1,2}(r)], c_2 \in [c_{2,1}(r), c_{2,2}(r)]\}, \]
\[ f_2(t, y; r) = \max\{c_1,u^2 + c_2|u \in [y_1(t; r), y_2(t; r)], c_1 \in [c_{1,1}(r), c_{1,2}(r)], c_2 \in [c_{2,1}(r), c_{2,2}(r)]\}. \]
By using the Runge-Kutta method of order 5 at $t_n$, $0 \leq n \leq N$
\[ k_{1,1}(t_n; r) = h(c_{1,1}(r),y_1^2(t_n; r) + c_{2,1}(r)), \]
\[ k_{1,2}(t_n; r) = h(c_{1,2}(r),y_2^2(t_n; r) + c_{2,2}(r)), \]
\[ k_{2,1}(t_n; r) = h(c_{1,1}(r),z_{1,1}^2(t_n; r) + c_{2,1}(r)), \]
\[ k_{2,2}(t_n; r) = h(c_{1,2}(r),z_{2,1}^2(t_n; r) + c_{2,2}(r)), \]
\[ k_{3,1}(t_n; r) = h(c_{1,1}(r),z_{2,1}^2(t_n; r) + c_{2,1}(r)), \]
\[ k_{3,2}(t_n; r) = h(c_{1,2}(r),z_{2,2}^2(t_n; r) + c_{2,2}(r)), \]
\[ k_{4,1}(t_n; r) = h(c_{1,1}(r),z_{3,1}^2(t_n; r) + c_{2,1}(r)), \]
\[ k_{4,2}(t_n; r) = h(c_{1,2}(r),z_{3,2}^2(t_n; r) + c_{2,2}(r)), \]
\[ k_{5,1}(t_n; r) = h(c_{1,1}(r),z_{4,1}^2(t_n; r) + c_{2,1}(r)), \]
\[ k_{5,2}(t_n; r) = h(c_{1,2}(r),z_{4,2}^2(t_n; r) + c_{2,2}(r)), \]
where
\[ z_{1,1}(t_n; r) = y_1(t_n; r) + \frac{1}{3}k_{1,1}(t_n; r), \]
\[ z_{1,2}(t_n; r) = y_2(t_n; r) + \frac{1}{3}k_{1,2}(t_n; r), \]
\[ z_{2,1}(t_n; r) = y_1(t_n; r) + \frac{1}{6}k_{1,1}(t_n; r) + \frac{1}{6}k_{2,1}(t_n; r), \]
\[ z_{2,2}(t_n; r) = y_2(t_n; r) + \frac{1}{6}k_{1,2}(t_n; r) + \frac{1}{6}k_{2,2}(t_n; r), \]
\[ z_{3,1}(t_n; r) = y_1(t_n; r) + \frac{1}{8}k_{1,1}(t_n; r) + \frac{3}{8}k_{3,1}(t_n; r), \]
\[ z_{3,2}(t_n; r) = y_2(t_n; r) + \frac{1}{8}k_{1,2}(t_n; r) + \frac{3}{8}k_{3,2}(t_n; r), \]
\[ z_{4,1}(t_n; r) = y_1(t_n; r) + \frac{1}{2}k_{1,1}(t_n; r) - \frac{3}{2}k_{3,1}(t_n; r) + 2k_{4,1}(t_n; r), \]
\[ z_{4,2}(t_n; r) = y_2(t_n; r) + \frac{1}{2}k_{1,2}(t_n; r) - \frac{3}{2}k_{3,2}(t_n; r) + 2k_{4,2}(t_n; r). \]
The exact and approximate solutions are shown in figure 2 at $t = 1.$
Table 2

<table>
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<th>r</th>
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<th>Exact Solution</th>
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Figure 2: h=0.5

6 Conclusion

In this work we have applied iterative solution of Runge-kutta Method of order five for numerical solution of fuzzy differential equations. It is clear that
the method introduced in Chapter with $O(h^5)$ performs better than *Improved Euler’s Method* with $O(h^2)$.

References


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