Split Operator Method for Parabolic Partial Differential Equation

Oduor Michael E. O.
Department of Mathematics and Applied Statistics
Maseno University, Box 333, Maseno, Kenya
oduor_okoya@yahoo.com

Nthiiri Joyce Kagendo
Department of Mathematics
Masinde Muliro University of Science and Technology
Box 190, Kakamega, Kenya

Abstract
In this paper we use the split operator technique to solve parabolic partial differential equation with seepage equation as a specific study case:

\[ u_t = \alpha u_{xx} + \beta u_{xt} \quad \text{subject to} \quad u(0,t) = 1 - e^{-t}, u(\infty,t) = 0, \quad u(x,0) = 0. \]

Keywords: Parabolic p.d.e., seepage equation, operator splitting

1 Introduction
Let \( u \) be a function of \( x \) and \( t \) where, \( x \) is a space variable and \( t \) is a time variable. Consider

\[ Lu = \sum_{S=1}^{t} I_{5} u, \quad S = 1,2,3,...,s, \quad (1.1) \]

where \( L \) is a differential operator in \( u \) and contains only partial derivatives.
with respect to space coordinates. This decomposition is referred to as Operator splitting technique, and is used to decompose parabolic partial differential equations into simpler sub problems and treat the sub problems individually using specialized numerical algorithms such as finite difference methods. Operator splitting is a widely used procedure in numerical solution of complex problems and helps simplify the parent problem thus help in achieving a remarkable resolution and accuracy in a very efficient manner; that is, even when very few splitting procedures are performed, See for instance [1]. Spatial differential operator appearing in a given parabolic partial differential equation can be split into a sum of different sub-operators of simpler forms with the corresponding equation being easier to solve using different numerical methods. Operator splitting technique increases grid/mesh points and also increases the order of accuracy of the scheme, see [1]. Operator splitting process makes it easier to theoretically investigate convergence which doing it for globally discretized unsplit problem is a very difficult task, see [2].

In this study, the basic idea is to split the parabolic partial differential equation of the form

\[ u_t = f(x,t,u^0_i^t_0,(u^0_i^t_0)_x, (u^0_i^t_0)_t) \quad \text{with} \quad (a \leq x \leq b) \times (0 < t \leq T); \quad p, q, g, i, j \in Z^+ \quad \text{and} \quad (1.2) \]

Where,

\[ u_x^t_0 = \frac{\partial^g_x u}{\partial^g_x t} \]

into smaller problems using operator splitting techniques.

Holden and Hvistendahl [8], applied the method of operator splitting on the generalized Korteweg-de Vries (KdV) equation,

\[ u_t + f(u)_x + \varepsilon u_{xxx} = 0 \quad \text{(1.3)} \]

where \( f(u)_x \) is the advection term and \( \varepsilon \) a constant by solving the nonlinear conservation law

\[ u_t + f(u)_x = 0 \quad \text{(1.4)} \]

and the linear dispersive equation

\[ u_t + \varepsilon u_{xxx} = 0 \quad \text{(1.5)} \]

sequentially. They analyzed convergent properties numerically by studying the effect of combining numerical methods for each of the simplified problems. In their work, the seepage parabolic partial differential equation was not analysed.
Operator splitting method has been studied by several authors, for example Crandall and Majda [4], Zlatev [12], Lanser et. al. [10], Marchuk [11], Gerisch and Verwer [5], Gourley & Morris [7], among others. Bii [3] solved the parabolic partial differential equation

\[ u_t = \alpha u_{xx} + \beta u_{xxt} \]  

subject to; \( u(0,t) = 1 - e^{-t}, \quad u(x,0) = 0, \quad u(x,t) = 0, \quad u(x,0) = 0. \) by finite difference technique. The scheme developed is of order \( O(k) + O(h)^2. \) In my knowledge, the seepage equation has not been solved using operator splitting method. Furthermore, error analysis using the method has not been done. This motivates this study a great deal.

2 Preliminaries

Consider the parabolic Equation (1.2). This equation can be decomposed into several sub problems, which are simpler and easier to solve than the parent problem. The \( S^{th} \) sub problem is given by

\[ u_t = \sum_{S=1}^S l_s = l_1 + l_2 + \ldots + l_s, \quad S = 1, 2, 3, \ldots, s. \]  

Consider the equation

\[ \frac{\partial u}{\partial t} = Lu, \]  

where \( Lu \) is a differential operator which contains partial derivatives with respect to the space coordinates \( x = x_1, x_2, \ldots, x_s; x \in R^s \) and whose coefficients which may either be constants or functions of both space and time variables. The symbol \( l_s \) represent the discrete operators. The above equation should be well posed; that is, a solution exists and is unique and depends on the given data. To obtain our solution operator, we use Taylor series expansion at a point \( (x, t + \Delta t) \)
\[ u(x, t + \Delta t) = u(x, t) + \Delta t \frac{\partial}{\partial t} u + \frac{\Delta t^2}{2!} \frac{\partial^2}{\partial t^2} u + \ldots \]
\[ = (1 + \Delta t \frac{\partial}{\partial t} + \frac{\Delta t^2}{2!} \frac{\partial^2}{\partial t^2} + \ldots) u \]
\[ = \left( \sum_{j=0}^{\infty} \frac{(\Delta t)^j}{j!} \frac{\partial^j}{\partial t^j} \right) u(x, t) \]
\[ = \left( e^{\Delta t L} \right) u \]
\[ = \left( e^{\Delta t L} \right) u. \quad \text{(2.3)} \]

In a discretized region, at the point \((x = m\Delta x, t = n\Delta t)\), where \(\Delta x\) and \(\Delta t\) are small increment in space and time respectively, we denote the numerical solution by \(U_{m,n} \). From equation (2.3), the numerical solution at the point \((m\Delta x, (n+1)\Delta t)\) is

\[ U_{m,n+1} = \left( e^{\Delta t L} \right) U_{m,n}. \quad \text{(2.4)} \]

Since \(L = \sum_{S=1}^{S} I_S\), we have

\[ U_{m,n+1} = \left( \prod_{i=1}^{S} e^{\Delta t l_i} \right) U_{m,n}. \quad \text{(2.5)} \]

Let \(S = 2\) Equation (2.1) becomes

\[ u_t = (l_1 + l_2) u \]
\[ \text{and the operator splitting method is} \]

\[ U_{m,n+1} = e^{\Delta t l_1} \left( e^{\Delta t l_2} U_{m,n} \right) \]
\[ = (1 + k l_1)(1 + k l_2) U_{m,n} + 0(k)^2 \]
\[ = U_{m,n} + k l_1 U_{m,n} + k l_2 U_{m,n} + k^2 l_1 l_2 U_{m,n}, \quad \text{(2.7)} \]
where \( k = \Delta t \).

Equation (2.7) can also be written as

\[
U_{m,n+1} = e^{\Delta l_2} \left( e^{\Delta l_1} U_{m,n} \right)
\]

\[
= (1 + kl_2)(1 + kl_1)U_{m,n} + 0(k)^2
\]

\[
= U_{m,n} + kl_1 U_{m,n} + kl_2 U_{m,n} + k^2 l_1 l_2 U_{m,n}.
\]  \hspace{1cm} (2.8)

Thus we see from Equations (2.7) and (2.8) that the operators commute.

The term \( O(k)^2 \) in (2.7) is called the order of the truncation error. The solution \( U_{m,n+1} \) is free from splitting error since \( e^{\Delta l_1}, e^{\Delta l_2}, \ldots \) commute, see [6]. All linear differential operators commute. However, if at least one of the differential operators is nonlinear, then they are non-commutative. For more on this see [6]

3 Numerical scheme using Split Operator Technique

Considering the parabolic partial differential Equation (1.6), where we denote our solution operator by,

\[
U_{m,n+1} = e^{\Delta l_i} , \hspace{1cm} (2.9)
\]

Where \( Lu = \sum_{S=1}^{2} l_s u \), we obtain after simple calculations

\[
U_{m,n+1} = \left[ 1 + \frac{2\beta}{h^2} \frac{6k\alpha \beta}{h^2} \right] + U_{m+1,n+1} \left[ \frac{4k\alpha \beta}{h^2} - \frac{\beta}{h^2} \right] + U_{m+2,n+1} \left[ -\frac{k\alpha \beta}{h^2} \right] + U_{m-1,n+1} \left[ \frac{4k\beta \alpha}{h^2} - \frac{\beta}{h^2} \right] + U_{m-2,n+1} \left[ \frac{2k\alpha}{h^2} + \frac{2\beta}{h^2} - \frac{6k\alpha \beta}{h^2} \right] + U_{m+1,n} \left[ \frac{k\alpha}{h^2} \frac{\beta}{h^2} + \frac{4k\alpha \beta}{h^2} \right] + U_{m+2,n} \left[ \frac{k\alpha}{h^2} \frac{\beta}{h^2} + \frac{4k\alpha \beta}{h^2} \right] - U_{m-1,n} \left[ \frac{k\beta \alpha}{h^2} \right]. \hspace{1cm} (2.11)
\]

The finite difference scheme for Equation (1.6) is
Comparing equation (2.11) with (2.12), we see that we have additional mesh points in (2.11) which include \( (U_{m+2,n}), (U_{m+2,n+1}), (U_{m-2,n}), (U_{m-2,n+1}) \)

Performing Taylor expansion in Equation (2.12) we obtain,

\[
\frac{\partial^3 u}{\partial x^2 \partial t} = \frac{1}{h^2 k} \left[ U_{m+1,n+1} + 2U_{m,n} + U_{m-1,n+1} - U_{m+1,n} - U_{m-1,n} - 2U_{m,n+1} \right] + O(k) + O(h)^2. \tag{3.1}
\]

Again from Taylor series expansion, Equation (2.11) becomes

\[
U_{m+2,n+1} + U_{m-2,n+1} + 4(U_{m+1,n} + U_{m-1,n}) + 6(U_{m,n+1} - U_{m,n}) - 4(U_{m-1,n+1} + U_{m-1,n+1}) = \\
\frac{h^4 k \frac{\partial^5}{\partial x^4 \partial t}}{6} + \frac{h^4 k \frac{\partial^7}{\partial x^6 \partial t}}{6} + \frac{h^4 k \frac{\partial^7}{\partial x^6 \partial t}}{6} + \frac{\partial^5}{\partial x^4 \partial t} \left[ \frac{U_{m+2,n+1} + U_{m-2,n+1} + 4(U_{m+1,n} + U_{m-1,n}) + 6(U_{m,n+1} - U_{m,n}) - 4(U_{m-1,n+1} + U_{m-1,n+1})}{4(U_{m-1,n+1} + U_{m-1,n+1})} \right] + O(k)^2 + O(h)^2. \tag{3.2}
\]

From Equation (3.2) we observe that splitting does not give room for error expansion but it increases the order of the method; that is, it enables one to increase the order of accuracy of time stepping in a straightforward manner from \( O(k) \) to \( O(k^2) \). In Equation (2.11) on letting \( \mu = \frac{1}{h^2}, r = \frac{k}{h^2}, \eta = \frac{k}{h^2}, \alpha = \beta = 1 \) and on rearranging becomes

\[
-r u_{m-2,n+1} + (4r - \mu) u_{m-1,n+1} + (1 - 2(\mu + 3r)) u_{m,n+1} + (4r - \mu) u_{m+1,n+1} - ru_{m+2,n+1} = ru_{m-2,n} + (\eta - \mu + 4r) u_{m-1,n} + (1 - 2(\eta - \mu + 3r)) u_{m,n} + (\eta - \mu + 4r) u_{m+1,n} + ru_{m+2,n}. \tag{3.3}
\]

Since our Equation (3.3) is implicit it will always be stable for all values of \( r \) since \( r \) has no restrictions.

From Equation (3.3), we can let various values for the parameters involved then solve using Mathematica to give graphical solutions. The following is a graph comparing the solutions when \( t=1(r=0.8, r=1) \).
3 Conclusion

We have analyzed split operator method for parabolic partial differential equations. The schemes developed using split operator method had added mesh points, that is \( (U_{m+2,n}), (U_{m+2,n+1}), (U_{m-2,n}), (U_{m-2,n+1}) \) as seen in Equation (2.11).

This leads to mesh refinement. The operator splitting method also improved the order of accuracy of the schemes in time variable from \( O(k) \) to \( O(k^2) \). The scheme in Equation (2.11) developed was implicit and is always stable for all values of \( r \) since \( r \) has no restrictions. We see that the smaller the mesh sizes, the more finely the results.

References


Received: August, 2011