Counting the Number of Spanning Trees in the Star Flower Planar Map

Abdulhafid Modabish and Mohamed El Marraki

Department of Computer Sciences, Faculty of Sciences
Mohammed V-Agdal University, P.O. Box 1014, Rabat, Morocco
hafizmod@yahoo.fr, marraki@fsr.ac.ma

Abstract

The number of spanning trees of a graph \( G \) is the total number of distinct spanning subgraphs of \( G \) that are trees (tree that visiting all the vertices of the graph \( G \)). Let \( C_n \) be a cycle with \( n \) vertices. The Star flower planar map is a simple graph \( G \) formed from a cycle \( C_n \) by adding a vertex adjacent to every edge of \( C_n \) and we connect this vertex with two end vertices of each edge of \( C_n \), i.e., we replace each edge of \( C_n \) by a triangulation. If there are \( k \) edges between every two vertices of each edge of the cycle \( C_n \), then we obtain the star flower planar map in the general case. In this work, we denote the star flower planar map by \( S_{n,k} \) where \( n \) is the number of triangles of the star flower planar map, \( k \) is the number of edges between each two vertices of each edge of the cycle \( C_n \), and derive the explicit formula for \( \tau(S_{n,k}) \) the number of spanning trees in \( S_{n,k} \) to be \( \tau(S_{n,k}) = 2kn(k+2)^{n-1}, \ n \geq 2. \)

Mathematics Subject Classification: 05C85, 05C30

Keywords: graphs, maps, spanning trees, star flower planar map

1 Introduction

An undirected graph \( G \) is a triple consisting of a vertex set \( V(G) \), an edge set \( E(G) \), and a relation that associates with each edge two vertices (not necessarily distinct) called its endpoints. A loop is an edge whose endpoints are equal. Multiple edges are edges having the same pair of endpoints. A graph \( G \) is connected if each pair of vertices in \( G \) belongs to a path; otherwise, \( G \) is disconnected. A graph which contains neither multiple edges nor loops is called a simple graph. A simple path is either an edge or a path \( p = v_0, v_1, v_2, ..., v_{n-1}, v_n \) such that \( \deg(v_i) = 2 \) for \( i = 1, 2, ..., n-1 \). A cycle is a path such that \( v_0 = v_n \). A planar graph \( G \) is a graph designed on the plane.
map $\mathcal{C}$ is a graph drawn on a surface $X$ or embedded into it (that is, a compact variety orientable 2-dimensional). A planar map is a map drawn on the plane. A tree is a connected graph without cycle. A plan tree is a tree designed in the plane.

The number of spanning trees in a graph (network) $G$, denoted by $\tau(G)$, is an important invariant of the graph (network). It is also an important measure of reliability of a network. A spanning tree in $G$ is a tree which has the same vertex set as $G$ (tree that passing through all the vertices of the graph $G$). Most research about the number of spanning trees is devoted to determining exact formulas for the number of spanning trees in many kinds of special graphs.

All graphs (maps) in this paper are undirected, connected, planar and without loops (because the loops don’t affect spanning trees, so we delete them before the computation). In this paper, we start by stating the general methods for counting the number of spanning trees in graphs; we then discuss our new results.

A famous and classic result on the study of $\tau(G)$ is the following theorem, known as the Matrix tree Theorem [3], [8].

The Laplacian matrix (also called Kirchhoff matrix) of a graph $G$ is defined as $L(G) = D(G) - A(G)$, where $D(G)$ and $A(G)$ are the degree matrix and the adjacency matrix of $G$, respectively.

**Theorem 1. (Matrix Tree Theorem)** For every connected graph $G$, If $L^*(G)$ is a matrix obtained by deleting row $i$ and column $j$ of the Laplacian matrix $L(G)$, then $\tau(G) = (-1)^{i+j} \det L^*(G)$ [8].

In [4], we used the determinant of Laplacian matrix of planar maps (Matrix Tree Theorem) to derive the explicit formula for calculating the number of spanning trees in a maximal planar map and deduce a formula for calculate the number of spanning trees of the crystal planar map.

However this theorem is not possible for large graphs, and various techniques are extended to find the number of spanning trees in different classes of graphs.

In [5], [6], we gave some methods to facilitate the calculation of the number of spanning trees of some particular planar maps, and apply these methods to give formulas to calculate the number of spanning trees of some special families of planar maps called n-Fan chains, n-Grid chains, n-Tent chains, n-Hexagonal chains, n-Home chains, n-Kite chains, n-Envelope chains,...etc. Other formulas are related to our work, as the Wheel, the Ladder and the Prism found in [1], [2], [7].
2 Main Results

The idea of this paper is how we calculate the number of spanning trees in the star flower planar graph (map) shown in Figure 1, when we do not want to use the determinant of Laplacian matrix of planar graphs (Matrix Tree Theorem).

This work has been extended to develop new techniques for the calculation of the number of spanning trees in a star flower planar map.

Let $C_n$ be a cycle with $n$ vertices, the number of spanning trees of this cycle is equal to the length of its cycle (the length of a cycle is the number of edges that form this cycle). The star flower planar map is a simple graph $G$ formed from a cycle $C_n$ by adding a vertex adjacent to every edge of $C_n$ and we connect this vertex with two end vertices of each edge of $C_n$, i.e., we replace each edge of $C_n$ by a triangulation (see Figure 2).

If there are $k$ edges (simple path in the general case) between every two vertices of each edge of the cycle $C_n$, then we obtain the star flower planar map shown
in Figure 3. In this work, we denote the star flower planar map by $S_{n,k}$ where $n$ is the number of triangles of the star flower planar map, $k$ is the number of edges (the length of simple path) between each two vertices of each edge of the cycle $C_n$; the star flower planar map has $n(k+1)$ vertices ($n$ vertices of degree 4 and $n + (k - 1)n$ of degree 2), $n(k+2)$ edges and $n+2$ faces ($n$ faces of degree $k+2$, one face of degree $nk$ and the other face of degree $2n$) (see Figure 3).

![Figure 3: The star flower planar map $S_{n,k}$](image)

**Example 1.** Here is an example of the star flower planar map $S_{2,k}$ with $n = 2$ (2 triangles) and $k$ edges, and the star flower planar map $S_{3,k}$ with $n = 3$ (3 triangles) and $k$ edges (see Figure 4).

![Figure 4: The star flower planar maps $S_{2,k}$ and $S_{3,k}$](image)

**Definition 1.** Let $C$ be a map that contains an edge $e = (v_1, v_2)$, we denote by $C - e$ the map obtained by deleting the edge $e = (v_1, v_2)$ and the map remains connected. We denote by $C.e$ or $C.v_1v_2$ the map obtained by deleting the edge $e$ and pasting the vertex $v_1$ with $v_2$ (contraction the two vertices $v_1$ and $v_2$).

**Theorem 2. (Cayley’s Theorem)** [8] Let $C$ be a map and $\tau(C)$ is the number of spanning trees of $C$. If $e = v_iv_{i+1} \in E(C)$ ($e$ is not a loop), then

$$\tau(C) = \tau(C - e) + \tau(C.e).$$
Now, we consider the case where the edge $e$ is replaced by a simple path $p = v_1, v_2, ..., v_k, v_{k+1}$ of length $k$ which connects $v_1$ with $v_{k+1}$.

**Definition 2.** Let $C$ be a map that contains a simple path $p$ formed by the vertices $v_1, v_2, ..., v_k, v_{k+1}$. We denote by $C - p$ the map obtained by deleting the simple path $p$ and the map remains connected. We denote by $C.p$ or $C.v_1v_{k+1}$ the map obtained by deleting the simple path $p$ and pasting two vertices $v_1$ and $v_{k+1}$.

**Example 2.** Here is an example of maps $C$, $C - p$ and $C.p$; see Figure. 5.

![Figure 5: An example of maps $C$, $C - p$ and $C.p$](image)

**Theorem 3.** *(Generalization of Cayley’s Theorem)* Let $C$ be a map, and let $p = v_1, v_2, ..., v_k, v_{k+1}$ be a simple path in the map $C$, then

$$\tau(C) = k\tau(C - p) + \tau(C.p),$$

where $k$ is the length of the path $p$ [6].

Now, we are interested by the maps of type $C = C_1 : C_2$, $C$ is a map such that if a path connects a vertex $v_k$ of $C_1$ and a vertex $v_l$ of $C_2$ must pass through $v_1$ or $v_2$; $C_1$ and $C_2$ have two common vertices $v_1, v_2$ (see Figure. 6).

![Figure 6: A map $C = C_1 : C_2$](image)

**Theorem 4.** [6] Let $C = C_1 : C_2$ be a map, $v_1$ and $v_2$ two vertices of the map $C$ which is formed by two maps $C_1$ and $C_2$ (see Figure. 6.), then

$$\tau(C) = \tau(C_1) \times \tau(C_2.v_1v_2) + \tau(C_1.v_1v_2) \times \tau(C_2).$$
3 Applications

Now, we derive the explicit formula for $\tau(S_{n,k})$ the number of spanning trees in $S_{n,k}$.

**Theorem 5.** The number of spanning trees of the star flower planar map $S_{n,k}$ (see Figure. 3) is given by the following formula:

$$\tau(S_{n,k}) = 2^{kn}(k+2)^{n-1}, \quad n \geq 2.$$  

**Proof:** We cut the one triangle as shown in Figure. 3, then we obtain the star flower planar map $S_{n-1,k}$ after cutting as follows (see Figure. 7):

![Figure 7: The star flower planar map $S_{n,k}$ after cutting](image)

and we apply Theorem 4, then we obtain:

$$\tau(S_{n,k}) = (k+2)^{n-1} \cdot 2k + \tau(S_{n-1,k}) \cdot (k+2)$$

$$(k+2)\tau(S_{n-1,k}) = 2k(k+2)^{n-1} + (k+2)^{2}\tau(S_{n-2,k})$$

$$(k+2)^2\tau(S_{n-2,k}) = 2k(k+2)^{n-1} + (k+2)^3\tau(S_{n-3,k})$$

$$(k+2)^3\tau(S_{n-3,k}) = 2k(k+2)^{n-1} + (k+2)^4\tau(S_{n-4,k})$$

$$(k+2)^{n-4}\tau(S_{4,k}) = 2k(k+2)^{n-1} + (k+2)^{n-3}\tau(S_{3,k})$$

$$(k+2)^{n-3}\tau(S_{3,k}) = 2k(k+2)^{n-1} + (k+2)^{n-2}\tau(S_{2,k})$$

$$(k+2)^{n-2}\tau(S_{2,k}) = 2k(k+2)^{n-1} + (k+2)^{n-1} \cdot 2k$$

By the sum of all the previous equations, we obtain:

$$\tau(S_{n,k}) = 2k(k+2)^{n-1}(n-1) + (k+2)^{n-2} \cdot 12k$$

We take $2k(k+2)^{n-1}$ as a factor, hence the result. \qed

**Particular cases:**
1) In the previous theorem, if we take $k = 1$ (see the star flower planar map $S_{n,k}$ shown in Figure 3), then we obtain the star flower planar map $S_{n,1}$ (see Figure. 8).

![Figure 8: The star flower planar map $S_{n,1}$](image)

**Corollary 1.** The number of spanning trees of the star flower planar map $S_{n,1}$ (see Figure. 8) is given by the following formula:

$$\tau(S_{n,1}) = 2n(3)^{n-1}, \quad n \geq 2.$$ 

2) In the previous theorem, if we take $k = 2$ (see the star flower planar map $S_{n,k}$ shown in Figure 3), then we obtain the star flower planar map $S_{n,2}$ (see Figure. 9).

![Figure 9: The star flower planar map $S_{n,2}$](image)

**Corollary 2.** The number of spanning trees of the star flower planar map $S_{n,2}$ (see Figure. 9) is given by the following formula:

$$\tau(S_{n,2}) = n(2)^{2n}, \quad n \geq 2.$$
Example 3. In the star flower planar map shown in Figure 1; we have \( n = 8 \) and \( k = 2 \), we apply Corollary 2 to find the number of spanning trees of this star flower planar map \( S_{8,2} \) (the idea of this work), then
\[
\tau(S_{8,2}) = 8(2)^{2\times8} = 524288.
\]

4 conclusion

In this paper, we have interested by calculating the number of spanning trees in the star flower planar map and gave some methods to calculate the number of spanning trees of some particular planar maps for that we have derived the explicit formula to calculate the number of spanning trees in the star flower planar map.

References


Received: December, 2011