Characterization of a Singulary Boundary Point of a Holomorphic Function

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Abstract
The theorems of Markov ([9], [10], [11]), Wynn ([14]) and De Montessus de Ballore ([7], [8]) show that one can do the analytic continuation of a power series outside of its disk of convergence. This can be very interesting to compute certain special functions using their series development, even if this one is divergent. Before applying such theorems, we must be sure that the series can be extended outside its disk of convergence. Let $f$ be a holomorphic function in an open connexe set $D$ of a complex plane. The analytical continuation of $f$ in a domain wider than $D$, depends in general on $f$ and the boundaries of $D$. The aim of this work is to try to characterize the boundary points of $D$, so that, the analytical continuation of $f$ in a domain wider than $D$, is possible or not.

Mathematics Subject Classification: 30E10, 32E30, 33F05, 11A55

Keywords: regular point, analytic continuation, algorithm

1 Introduction

The expansion of a function $f$ in power series

$$f(x) = \sum_{k=0}^{+\infty} a_k x^k,$$

(1)
is generally not used in practice to calculate the values of $f$, because equality (1) is valid only for values of $x$ in the interior of the disc of convergence of the entire series:

$$
\sum_{n=0}^{+\infty} a_n x^n.
$$

On the other hand, the convergence of the partial sums

$$
s_n(x) = \sum_{k=0}^{n} a_k x^k
$$

for $x$ in the interior of the disk of convergence, to $f(x)$, is generally slow. In other words, we should calculate a large number of terms in the series $\sum_{n=0}^{+\infty} a_n x^n$, to obtain a good approximation of $f(x)$. To overcome this problem, we can use the expansion of $f$ in continuous fraction, or use sequences of Padé approximants associated to the series $\sum_{n=0}^{+\infty} a_n x^n$. One can also accelerate the convergence of the sequence of partial sums $s_n(x)$ of the series $\sum_{n=0}^{+\infty} a_n x^n$, by using for example the $\varepsilon$-Algorithm. In reality, these three methods (expansion of $f$ in continuous fraction, Padé approximation and $\varepsilon$-Algorithm) are related, and using one of the three methods, amounts to using one of the two others. Let us give some short concepts on the $\varepsilon$-Algorithm. For details see ([2], [3], [4]).

The $\varepsilon$-Algorithm is due to Peter Wynn ([14]). Being given a sequence $\{s_k\}_{k \in \mathbb{N}}$, the Epsilon Algorithm computes quantities $\varepsilon_{j}^{(k)}(j, k = 0, 1, \ldots)$, with two indices, as follows:

$$
\begin{align*}
\varepsilon_{-1}^{(k)} &= 0 \\
\varepsilon_{0}^{(k)} &= s_k \\
\varepsilon_{j+1}^{(k)} &= \varepsilon_{j}^{(k+1)} + \frac{1}{\varepsilon_{j}^{(k+1)} - \varepsilon_{j}^{(k)}} & k = 0, 1, \\
& & j, k = 0, 1, \ldots
\end{align*}
$$

These numbers can be placed in a double entry table as follows:
The Epsilon algorithm connects four quantities belonging to the four nodes of a Losange.

\[ \epsilon_{-1}^{(0)} = 0 \]
\[ \epsilon_{-1}^{(1)} = 0 \]
\[ \epsilon_{-1}^{(2)} = 0 \]
\[ \epsilon_{-1}^{(3)} = 0 \]
\[ \epsilon_{-1}^{(4)} = 0 \]
\[ \epsilon_{-1}^{(5)} = 0 \]
\[ \epsilon_{-1}^{(6)} = 0 \]
\[ \epsilon_{-1}^{(7)} = 0 \]

\[ \epsilon^{(0)}_0 = x_0 \]
\[ \epsilon^{(1)}_0 = x_1 \]
\[ \epsilon^{(2)}_0 = x_2 \]
\[ \epsilon^{(3)}_0 = x_3 \]
\[ \epsilon^{(4)}_0 = x_4 \]
\[ \epsilon^{(5)}_0 = x_5 \]
\[ \epsilon^{(6)}_0 = x_6 \]

We can notice that to compute the quantity \( \epsilon^{(k)}_{j+1} \), we need the numbers \( \epsilon^{(k+1)}_{j-1} \), \( \epsilon^{(k+1)}_{j} \) and \( \epsilon^{(k)}_{j} \).

The theorems, numerical examples cited in this paragraph can be found in [2], [3], [4].

Let us consider the series associated with \( \log (1 + x) \)

\[ \log (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... \]
Its convergence radius is equal to 1. For $x = 2$, we have: $\log 3 = 1, 098612288668110...$

<table>
<thead>
<tr>
<th>Suite des sommes partielles de $\log(3)$</th>
<th>$\varepsilon$-algorithme</th>
</tr>
</thead>
<tbody>
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<tr>
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</tr>
<tr>
<td>.2395765779986066</td>
<td>.1098612288668233+001</td>
</tr>
</tbody>
</table>
We can see, that the initial sequence (partial sum of the series of log(3)) diverges quickly, while the $\varepsilon$-Algorithm converges rapidly to log(3). The $\varepsilon$-Algorithm makes it possible to compute log(3) outside the disc of convergence for the series: $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$. It also allows to accelerate the convergence.

Having said that, we are going now to recall the famous theorem of Montessus de Ballore, expressed in terms of the $\varepsilon$-Algorithm. For the relation between the Padé approximants and the $\varepsilon$-Algorithm, see [2], [3] and [4].

**Théorème 1** (R. De Montessus de Ballore [7], [8], [3])

Let $f(x) = \sum_{i=0}^{+\infty} c_i x^i$ be a series and let $R = \lim_{n \to \infty} |c_n|^{1/2}$ is its convergence radius. If $|x| \geq R$ and if $f$ has $p$ poles $x_1, \ldots, x_m$ computed with their multiplicity on and in the circle $|x| = a$ (we suppose that there is no other singularities). Then, the sequence $\{\varepsilon_{2p}^{(n)}\}$ (for fixed $p$) converges to the value of $f(x)$, obtained by the analytic continuation of the sum of the series outside the circle: $|x| = |x_1|$. ($x \neq x_1, \ldots, x \neq x_m$, $|x| \leq a$, $x_1$ is the pole of the smaller modulus).

To illustrate this theorem, let us consider the following example:

$$f(x) = e^x + \frac{1}{(1-x)^3} = \sum_{i=0}^{\infty} \left(\frac{1}{i!} + a_i\right) x^i$$

with

$$a_0 = 1 \quad \text{and} \quad a_i = a_{i-1} + i + 1 \quad \text{for} \quad i = 1, 2, \ldots$$

$f$ has a pole of multiplicity 3 ($x = 1$). Thus, we have $p = 3$ in the theorem of De Montessus de Ballore. The sequence $\{\varepsilon_{6}^{(n)}\}$ must converge to $f(x)$ even outside the disk of convergence. i.e. even when $|x| > 1$.

For $x = 1.5$ we have:

$$f(x) = -3.51831092966193527$$

By applying the $\varepsilon$-Algorithm to the partial sums of this series when $x = 1.5$, ...
we obtain the following table:

<table>
<thead>
<tr>
<th>$\varepsilon_{0}^{(n)}$</th>
<th>$\varepsilon_{6}^{(n)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.2000000000000+001</td>
<td></td>
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<tr>
<td>.8000000000000+001</td>
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<tr>
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<td>15047056884077+005</td>
<td>-.35661701000621+001</td>
</tr>
<tr>
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<td>-.35258309517364+001</td>
</tr>
<tr>
<td>.47289021850199+005</td>
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</tr>
<tr>
<td>.82320533080672+005</td>
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<tr>
<td>.11571858237568+007</td>
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</tr>
</tbody>
</table>

Other theorems give results similar to that of De Montessus de Ballore. Let us mention the theorems of Markov ([9], [10]) and Wynn ([14]). These three theorems show that we can express the analytic continuation of a power series outside of its disk of convergence. This can be very interesting to compute certain special functions using their series development, even if this one is divergent. Before applying such theorems, We must be sure that the series is analytically prolonged outside its disk of convergence. The series: $f(z) = \sum_{n=0}^{+\infty} z^n$ admits the open ball : $\{ z : |z| < 1 \}$ as natural domain; which means that there exist no analytical continuation of $f$ in a domain wider that $\{ z/|z| < 1 \}$ (see [12], [13]) .
For $x = 0.85$, we obtain

\[
\begin{align*}
\{ \varepsilon_0^{(n)} \} &= \{ s_n(0, 85) \} & \{ \varepsilon_2^{(n)} \} & \{ \varepsilon_4^{(n)} \} \\
1 & & & \\
2.37 & 2.79 & 2.67 \\
2.60 & 2.71 & 2.695 \\
2.67 & 2.70 & 2.6980 \\
2.695 & 2.6985 & 2.69831 \\
2.6979 & 2.698346 & 2.699329 \\
2.6983 & 2.698331 & 2.69983035 \\
2.698328 & 2.69833043 & 2.6998303913
\end{align*}
\]

For $x = 1.05$ we obtain:

\[
\begin{align*}
\{ \varepsilon_0^{(n)} \} &= \{ s_n(1, 05) \} & \{ \varepsilon_2^{(n)} \} & \{ \varepsilon_4^{(n)} \} \\
1 & & & \\
2.05 & -5.66 & & \\
.3.36 & -2.35 & 11.13 & \\
4.82 & -0.54 & -36.51 & \\
6.99 & 0.86 & -5.88 & \\
10.38 & 2.23 & -2.11 & \\
16.18 & 3.85 & -0.18 & \\
27.09 & 6.05 & & \\
49.80 & & & \\
\end{align*}
\]

We can see clearly, that for $x = 0.85$ which belongs to the disc of convergence of the series $\sum_{n=0}^{+\infty} z^n$, the three sequences $\{ s_n(0.85) \}$, $\{ \varepsilon_2^{(n)} \}$ and $\{ \varepsilon_4^{(n)} \}$ converge towards the same limit. On the contrary, for $x = 1.05$ which is located outside the disk of convergence of the series $\sum_{n=0}^{+\infty} z^n$, the three sequences $\{ s_n(1.5) \}$, $\{ \varepsilon_2^{(n)} \}$ and $\{ \varepsilon_4^{(n)} \}$ diverge.

Let $f$ be a holomorphic function in an open connexe set $D$ of a complex plane. The analytical continuation of $f$ in a domain wider than $D$ depends in general on $f$ and the boundaries of $D$.

The aim of this work is to try to characterize the boudary points of $D$ so that the analytical continuation of $f$ in a domain wider than $D$ be possible or not.

In the chapter 3, we will study the particular case where $D$ is an open disc. We will try to generalize this result to the case where $D$ is an open connexe set, presenting some geometrical properties.
2 Regularity, Singularity, Analytic continuation

Definition 1 ([5], [6], [12], [13])

Let $D$ be an open connexe set of $\mathbb{C}$ and $f$ be a function belonging to the space of holomorphic functions in $D$ which we denoted by $H(D)$, and $\mu$ a boundary point of $D$. We say that $\mu$ is a regular point for $f$, if it exists an open disc of center $\mu$ and radius $r$, denoted by $B(\mu, r)$, and $g \in H((B\mu, r))$ such that:

$$f(z) = g(z) \quad \text{for} \quad z \in D \cap B(\mu, r)$$

$\mu$ is singular for $f$, if it is not regular.

Remark 1

If $\mu$ is regular for $f$, then $f$ is analytically extended from $D$ to $D \cup B(\mu, r)$ (see [12], [13]).

Definition 2

If all the boundary points of $D$ are singular for $f$, then the boundary of $D$ is called the natural boundary of $f$, and $D$ is the natural domain of $f$.

Remark 2

If the boundary of $D$ is the natural boundary of $f$, then $f$ does not have any analytical continuation, in a domain bigger than $D$. This is the case of the series $f(z) = \sum_{n=0}^{\infty} z^n$, which admits the open unity disc: $\{z \in \mathbb{C} : |z| < 1\}$ as a natural domain of $f$.

Remark 3

If $\mu$ is an isolated boundary point of $D$, then the Laurent expansion of $f$, characterizes completely the regular or singular property of the point $\mu$ (see [5], [6], [12], [13]).

3 Characterization of a non-isolated singular boundary point of a holomorphic function in the case of a disc

Denote by $Fr(D)$ the boundary of the domain $D$ in $\mathbb{C}$ (we assume that $D$ is not empty and not equal to $\mathbb{C}$),

$$B(a, r) = \{z \in \mathbb{C} / |z - a| < r\}$$
$$\overline{B}(a, r) = \{ z \in \mathbb{C} / |z - a| \leq r \}$$

$$]z_1, z_2[ = \{(1-t)z_1 + tz_2 / 0 < t < 1\}, \ z_1 \in \mathbb{C}, \ z_2 \in \mathbb{C}$$

Let $\mu \in Fr(B(0, R))$, $f \in H(B(0, R))$, $z_1 \in ]0, \mu[$. Denote by $\Phi(z_1)$, the convergence radius of the entire series:

$$\sum_{k=0}^{+\infty} \frac{f^{(n)}(z_1)}{n!}(z-z_1)^n.$$  \hspace{1cm} (3)

It is known (see[1], [5], [6]) that, if $\mu$ is singular for $f$, then:

$$\Phi(z_1) = R - |z_1|$$

We propose to show that the reciprocal is also true.

**Theorem 2**

Let $f \in H(B(0, R))$, $\mu \in Fr(B(0, R))$, $z_1 \in ]0, \mu[$ and $\Phi(z_1)$ is the convergence radius of the entire series $\sum_{n=0}^{+\infty} \frac{f^{(n)}(z_1)}{n!}(z-z_1)^n$.

If

$$\Phi(z_1) = R - |z_1|$$

then $\mu$ est singular for $f$.

**Proof of theorem 2**

Effectively. Suppose that $\mu$ is a regular point for $f$, then, $\exists B(\mu, r) : r > 0$ and $g \in H(B(\mu, r))$ such that: $f(z) = g(z) : z \in B(\mu, r) \cap B(0, R)$. Define $F$ as follows:

$$F : B(0, R) \cup B(\mu, r) \to \mathbb{C}$$

with

$$F(z) = f(z) : \ z \in B(0, R)$$
$$F(z) = g(z) : \ z \in B(\mu, R)$$

$F$ is well defined and $F \in H(B(0, R) \cup B(\mu, r))$. then $F$ admits the following expansion:

$$F(z) = \sum_{n=0}^{+\infty} \frac{F^{(n)}(z_1)}{(n!)}(z-z_1)^n$$  \hspace{1cm} (4)
Te series (4) is convergent at least in the open disk $B(z_1, d)$ where:

$$d = \inf |z_1 - s| : s \in Fr [B(0, R) \cup (\mu, r)]$$

(5)

It is simple to see that:

$$d > R - |z_1|$$

(6)

Let us note by $\gamma$, the following circle:

$$\gamma : [0, 2\pi] \to \mathbb{C} : t \to \gamma(t) = z_1 + \lambda e^{it}, \ 0 < \lambda < R - |z_1|.$$ We have

$$F^{(n)}(z_1) = \frac{n!}{2i\pi} \int_{\gamma} \frac{F(u)}{(u - z_1)^{n+1}} du$$

$$= \frac{n!}{2i\pi} \int_{\gamma} \frac{f(u)}{(u - z_1)^{n+1}} du = f^{(n)}(z_1),$$

which contradicts the fact that $\Phi(Z_1) = R - |z_1|$. ■

4 Generalization

Let us denote by $T(\mathbb{C})$ the family of connex, non empty open sets of $\mathbb{C}$, possessing the following geometrical properties:

**Definition 3** Let $D$ be a non empty open and connex set of $\mathbb{C}$, $(D \neq \mathbb{C})$. $D \in T(\mathbb{C})$ if and only if:

$$\forall \beta \in Fr(D), \ \exists B(z(\beta), r(\beta)) \ such \ that \ \beta \in \overline{B}(z(\beta), r(\beta))$$

and

$$[\overline{B}(z(\beta), r(\beta)) \setminus \{\beta\}] \subset D.$$ With these notations we obtain:

**Theorem 3**

Let $D \in T(\mathbb{C})$, $f \in H(D)$, $\beta \in Fr(D)$ not isolated, $z_1 \in [z(\beta), \beta]$ arbitrary. Denote by $\Phi(z_1)$, the convergence radius of the entire series:
Characterization of a singulary boundary point

\[ \sum_{k=0}^{+\infty} \frac{f^{(n)}(z_1)}{n!} (z - z_1)^n \quad (7) \]

Then \( \beta \) is singular for \( f \) if and only if:

\[ \tilde{\Phi}(z_1) = r(\beta) - |z(\beta) - z_1| \quad (8) \]

**Proof of Theorem 3:**

a) let us show that if:

\[ \tilde{\Phi}(z_1) = r(\beta) - |z(\beta) - z_1| \]

Then \( \beta \) is singular for \( f \). Effectively, let us suppose that \( \beta \) is regular. Then:

\( \exists B(\beta, r) \) and \( g \in H(B(\beta, r)) \)

such that

\( f(z) = g(z) \) for \( z \in B(\beta, r) \cap D \)

Let us consider the analytical continuation \( F \) of \( f \), from \( D \) to \( D \cup B(\beta, r) \),

\[ F(z) = \begin{cases} f(z) & \text{if } z \in D \\ g(z) & \text{if } z \in B(\beta, r) \end{cases} \]

It is simple to prove that

\( F \in H(D \cup B(\beta, r)) \).

Let us consider the entire series:

\[ \sum_{k=0}^{+\infty} \frac{F^{(n)}(z_1)}{n!} (z - z_1)^n \quad (9) \]

The series (9) is convergent at least in \( B(z_1, d_1) \), where:

\[ d_1 = \inf \{|z_1 - s| : s \in Fr(D \cup B(\beta, r))\} \quad (10) \]

We have

\[ d_1 > \tilde{\Phi}(z_1) = r(\beta) - |z(\beta) - z_1| \quad (11) \]
This is due owing to the fact that:

$$\forall \alpha \in \{ B(z(\beta), r(\beta)) \setminus \{\beta\} : |z_1 - \alpha| > |z_1 - \beta|$$

and

$$\overline{B}(z(\beta), r(\beta)) \subset D \cup B(\beta, r) \quad \text{and} \quad \beta \in B(\beta, r).$$

Let us now consider the circle $\tilde{\gamma}$:

$$\tilde{\gamma} : [0, 2\pi] \to \mathbb{C}$$

$$t \to z_1 + \eta e^{it}; \quad \text{such that} : \quad 0 < \eta < r(\beta) - |z(\beta) - z_1|.$$ 

We have of course:

$$\frac{F^{(n)}(z_1)}{n!} = \frac{1}{2i\pi} \int_{\tilde{\gamma}} \frac{F(u)}{(u-z_1)^{n+1}} \, du$$

$$= \frac{1}{2i\pi} \int_{\tilde{\gamma}} \frac{f(u)}{(u-z_1)^{n+1}} \, du = \frac{f^{(n)}(z_1)}{n!}.$$ 

Thus the series $\sum_{n=0}^{+\infty} \frac{f^{(n)}(z_1)}{n!} (z - z_1)^n$ is convergent in $B(z_1, d_1)$. This is in contradiction with the fact that $d_1 > \tilde{\Phi}(z_1)$ (see [1], [5], [6], [12], [13]).

b) If $\beta$ is singular for $f$, then $\tilde{\Phi}(z_1) = r(\beta) - |z(\beta) - z_1|$. This is obvious (see [1], [5], [6], [12], [13]).

**Corrollary 1**

Let $D \in T(\mathbb{C})$, $f \in H(D)$, $\beta \in Fr(D)$ not isolated, $z_1 \in ]z(\beta), \beta[$ arbitrary and $\tilde{\Phi}(z_1)$ is the convergence radius of the entire series:

$$\sum_{n=0}^{+\infty} \frac{f^{(n)}(z_1)}{n!} (z - z_1)^n$$

Then $\beta$ is regular for $f$, if and only if:

$$\tilde{\Phi}(z_1) > r(\beta) - |z(\beta) - z_1|$$

**Proof of corrolary 1**

It is obvious, by noticing that, we always have:

$$\tilde{\Phi}(z_1) > r(\beta) - |z(\beta) - z_1| \quad (\text{see} \ [12], \ [13] \ \text{and theorem3})$$
References


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