On the Twisted $q$-Genocchi Numbers and Polynomials with Weak Weight $\alpha$

C. S. Ryoo

Department of Mathematics
Hannam University, Daejeon 306-791, Korea
ryoocs@hnu.kr

J. Y. Kang

Department of Mathematics
Hannam University, Daejeon 306-791, Korea

Abstract

In this paper we construct a new type of twisted $q$-Genocchi numbers and polynomials with weak weight $\alpha$. We investigate some properties which are related to twisted $q$-Genocchi numbers $G^{(\alpha)}_{n,q,w}$ and polynomials $G^{(\alpha)}_{n,q,w}(x)$ with weak weight $\alpha$.

Mathematics Subject Classification: 11B68, 11S40, 11S80

Keywords: Genocchi numbers and polynomials, twisted $q$-Genocchi numbers and polynomials, twisted $q$-Genocchi numbers and polynomials with weak weight $\alpha$

1 Introduction

The Genocchi numbers and polynomials possess many interesting properties and arising in many areas of mathematics and physics. Recently, many mathematicians have studied in the area of the Genocchi numbers and polynomials (see [1-8]). In this paper, we construct a new type of twisted $q$-Genocchi numbers $G^{(\alpha)}_{n,q,w}$ and polynomials $G^{(\alpha)}_{n,q,w}(x)$ weak weight $\alpha$.

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of $p$-adic rational integers, $\mathbb{Q}_p$ denotes the field of $p$-adic rational numbers, $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{C}$ denotes the set of complex numbers, and
\( \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \). Let \( \nu_p \) be the normalized exponential valuation of \( \mathbb{C}_p \) with \( |p|_p = p^{-\nu_p(p)} = p^{-1} \). When one talks of \( q \)-extension, \( q \) is considered in many ways such as an indeterminate, a complex number \( q \in \mathbb{C} \), or \( p \)-adic number \( q \in \mathbb{C}_p \). If \( q \in \mathbb{C} \) one normally assume that \( |q| < 1 \). If \( q \in \mathbb{C}_p \), we normally assume that \( |q - 1|_p < p^{-\frac{1}{m}} \) so that \( q^x = \exp(x \log q) \) for \( |x|_p \leq 1 \). Throughout this paper we use the notation:

\[
[x]_q = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q} \quad \text{(cf. [1-8])}.
\]

Hence, \( \lim_{q \to 1} [x] = x \) for any \( x \) with \( |x|_p \leq 1 \) in the present \( p \)-adic case. For \( g \in UD(\mathbb{Z}_p) = \{ g|g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\} \), the fermionic \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) is defined by Kim as follows:

\[
I_{-q}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} g(x)(-q)^x \quad \text{(cf. [3-6])}.
\] (1.1)

If we take \( g_1(x) = g(x + 1) \) in (1.1), then we easily see that

\[
qI_{-q}(g_1) + I_{-q}(g) = [2]_q g(0).
\] (1.2)

From (1.2), we obtain

\[
q^nI_{-q}(g_n) + (-1)^{n-1}I_{-q}(g) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l g(l),
\] (1.3)

where \( g_n(x) = g(x + n) \) (cf. [3-6]).

As well known definition, the Genocchi polynomials are defined by

\[
F(t) = \frac{2t}{e^t + 1} = e^{Gt} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!},
\]

\[
F(t, x) = \frac{2t}{e^t + 1} e^{xt} = e^{G(x)t} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!},
\]

with the usual convention of replacing \( G^n(x) \) by \( G_n(x) \). In the special case, \( x = 0 \), \( G_n(0) = G_n \) are called the \( n \)-th Genocchi numbers (cf. [1-8]).

Our aim in this paper is to define twisted \( q \)-Genocchi numbers \( G^{(\alpha)}_{n,q,w} \) and polynomials \( G^{(\alpha)}_{n,q,w}(x) \) with weak weight \( \alpha \). Some interesting results and relationships are obtained. We also derive the existence of a specific interpolation function which interpolate twisted \( q \)-Genocchi numbers \( G^{(\alpha)}_{n,q,w} \) and polynomials \( G^{(\alpha)}_{n,q,w}(x) \) with weak weight \( \alpha \) at negative integers.
Twisted $q$-Genocchi numbers and polynomials with weak weight $\alpha$

In this section, we introduce the twisted $q$-Genocchi numbers $G^{(\alpha)}_{n,q,w}$ and polynomials $G^{(\alpha)}_{n,q,w}(x)$ with weak weight $\alpha$ and investigate their properties. We also find generating functions of twisted $q$-Genocchi numbers $G^{(\alpha)}_{n,q,w}$ and polynomials $G^{(\alpha)}_{n,q,w}(x)$ with weak weight $\alpha$. Let $T_p = \bigcup_{N \geq 1} C_{p^N} = \lim_{N \to \infty} C_{p^N}$, where $C_{p^N} = \{ w | w^{p^N} = 1 \}$ is the cyclic group of order $p^N$. For $w \in T_p$, we denote by $\phi_w : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \mapsto w^x$.

For $\alpha \in \mathbb{Z}$ and $q \in \mathbb{C}_p$ with $|1 - q|_p \leq 1$, twisted $q$-Genocchi numbers $G^{(\alpha)}_{n,q,w}$ are defined by

$$G^{(\alpha)}_{n,q,w} = n \int_{\mathbb{Z}_p} \phi_w(x)[x]_q^{n-1}d\mu_{-q^n}(x). \quad (2.1)$$

By using $p$-adic $q$-integral on $\mathbb{Z}_p$, we obtain,

$$n \int_{\mathbb{Z}_p} \phi_w(x)[x]_q^{n-1}d\mu_{-q^n}(x) = n \lim_{N \to \infty} \frac{1}{[p^N]_q^{-\alpha}} \sum_{x=0}^{p^N-1} [x]_q^{n-1}w^x(-q^\alpha)^x$$

$$= n[2]_q^{\alpha} \left( \frac{1}{1-q} \right)^{n-1} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l \frac{1}{1+wq^{\alpha+l}}$$

$$= n[2]_q^{\alpha} \sum_{m=0}^{\infty} (-1)^m w^m q^{\alpha m} [m]_q^{n-1}. \quad (2.2)$$

By (2.1), we have

$$G^{(\alpha)}_{n,q,w} = n[2]_q^{\alpha} \left( \frac{1}{1-q} \right)^{n-1} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l \frac{1}{1+wq^{\alpha+l}}$$

$$= n[2]_q^{\alpha} \sum_{m=0}^{\infty} (-1)^m q^{\alpha m} w^m [m]_q^{n-1}$$

We set

$$F^{(\alpha)}_{q,w}(t) = \sum_{n=0}^{\infty} G^{(\alpha)}_{n,q,w} \frac{t^n}{n!}.$$
By using above equation and (2.2), we have
\[
F_{q,w}^{(\alpha)}(t) = \sum_{n=0}^{\infty} G_{n,q,w}^{(\alpha)} \frac{t^n}{n!} \\
= [2]_{q^\alpha} \sum_{n=0}^{\infty} \left( n \left( \frac{1}{1-q} \right) \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l \frac{1}{1+wq^{\alpha+l}} \right) \frac{t^n}{n!} \\
= t[2]_{q^\alpha} \sum_{m=0}^{\infty} (-1)^m q^{\alpha m} w^m e^{[m]q^t}. \quad (2.3)
\]

Thus twisted \( q \)-Genocchi numbers \( G_{n,q,w}^{(\alpha)} \) with weak weight \( \alpha \) are defined by means of the generating function
\[
F_{q,w}^{(\alpha)}(t) = t[2]_{q^\alpha} \sum_{m=0}^{\infty} (-1)^m q^{\alpha m} w^m e^{[m]q^t}. \quad (2.4)
\]

By using (2.1), we have
\[
\sum_{n=0}^{\infty} G_{n,q,w}^{(\alpha)} \frac{t^n}{n!} = \sum_{n=0}^{\infty} n \int_{\mathbb{Z}_p} \phi_w(x) [x]_{q^\alpha}^{n-1} d\mu_{-q^\alpha}(x) \frac{t^n}{n!} \\
= t \int_{\mathbb{Z}_p} \phi_w(x) e^{[x]q^t} d\mu_{-q^\alpha}(x). \quad (2.5)
\]

By (2.3), (2.5), we have
\[
t \int_{\mathbb{Z}_p} \phi_w(x) e^{[x]q^t} d\mu_{-q^\alpha}(x) = t[2]_{q^\alpha} \sum_{m=0}^{\infty} (-1)^m q^{\alpha m} w^m e^{[m]q^t}.
\]

Next, we introduce twisted \( q \)-Genocchi polynomials \( G_{n,q,w}^{(\alpha)}(x) \) with weak weight \( \alpha \). The twisted \( q \)-Genocchi polynomials \( G_{n,q,w}^{(\alpha)}(x) \) with weak weight \( \alpha \) are defined by
\[
G_{n+1,q,w}^{(\alpha)}(x) = (n+1) \int_{\mathbb{Z}_p} \phi_w(y) [x+y]_{q^\alpha}^n d\mu_{-q^\alpha}(y). \quad (2.6)
\]

By using \( p \)-adic \( q \)-integral, we obtain
\[
G_{n+1,q,w}^{(\alpha)}(x) = (n+1)[2]_{q^\alpha} \left( \frac{1}{1-q} \right) \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{xl} \frac{1}{1+wq^{\alpha+l}}. \quad (2.7)
\]

We set
\[
F_{q,w}^{(\alpha)}(t, x) = \sum_{n=0}^{\infty} G_{n,q,w}^{(\alpha)}(x) \frac{t^n}{n!}. \quad (2.8)
\]
By using (2.7) and (2.8), we obtain

\[ F_{q,w}^{(\alpha)}(t, x) = \sum_{n=0}^{\infty} G_{n,q,w}^{(\alpha)}(x) \frac{t^n}{n!} = t[2]_{q^\alpha} \sum_{m=0}^{\infty} (-1)^m q^{\alpha m} w^m e^{[m+x]_q t}. \] (2.9)

Observe that, if \( q \to 1 \) and \( w = 1 \), then \( F_{q,w}^{(\alpha)}(t, x) \to F(t, x) \) and \( F_{q,w}^{(\alpha)}(t) \to F(t) \).

Since \([x + y]_q = [x]_q + q^x[y]_q\), we easily obtain that

\[ G_{n+1,q,w}^{(\alpha)}(x) = (n + 1) \int_{\mathbb{Z}_q} \phi_w(y)[x + y]_q^n d\mu_{-q^\alpha}(y) \]

\[ = q^{-x} \sum_{k=0}^{n+1} \binom{n+1}{k} [x]_q^{n+1-k} q^{xk} G_{k,q,w}^{(\alpha)} \]

\[ = q^{-x} ([x]_q + q^x G_{q,w}^{(\alpha)})^{n+1} \]

\[ = (n + 1)[2]_{q^\alpha} \sum_{m=0}^{\infty} (-1)^m q^{\alpha m} w^m [x + m]_q^n \] (2.10)

Observe that, if \( q \to 1 \) and \( w = 1 \), then \( G_{n,q,w}^{(\alpha)} \to G_n \) and \( G_{n,q,w}^{(\alpha)}(x) \to G_n(x) \).

By (2.7), we have the following theorem.

**Theorem 2.1** (complement relation). For \( n \in \mathbb{Z}_+ \), we have

\[ G_{n,q^{-1},w^{-1}}^{(\alpha)}(1 - x) = (-1)^{n-1} q^{n-1} w G_{n,q,w}^{(\alpha)}(x) \]

By (1.3), (2.1), and (2.6), we easily see that

\[ m[2]_{q^\alpha} \sum_{l=0}^{n-1} (-1)^{n-1-l} q^{\alpha l}[l]_q^{m-1} = q^m G_{m,q}^{(\alpha)}(n) + (-1)^{n-1} G_{m,q}^{(\alpha)}. \]

Hence, we have the following theorem.

**Theorem 2.2** For any positive integer \( m \equiv 1 \pmod{2} \), we have

\[ G_{n,q,w}^{(\alpha)}(x) = \frac{[2]_{q^\alpha}}{[2]_{q^\alpha}} |m|_q^{n-1} \sum_{i=0}^{m-1} (-1)^{i} q^{\alpha i} w^i G_{n,q,m,w}^{(\alpha)} \left( \frac{i + x}{m} \right), \quad n \in \mathbb{Z}_+. \]

By (1.3), (2.1), and (2.6), we easily see that

\[ m[2]_{q^\alpha} \sum_{l=0}^{n-1} (-1)^{n-1-l} q^{\alpha l}[l]_q^{m-1} = q^m w^n G_{m,q,w}^{(\alpha)}(n) + (-1)^{n-1} G_{m,q,w}^{(\alpha)}. \]

Hence, we have the following theorem.
Theorem 2.3 Let $m \in \mathbb{Z}_+$. If $n \equiv 0 \pmod{2}$, then
\[
q^{\alpha n} w^n G_{m,q,w}^{(\alpha)}(n) - G_{m,q,w}^{(\alpha)} = m[2]_{q^\alpha} \sum_{l=0}^{n-1} (-1)^{l+1} q^{\alpha l} w^l[l]_q^{m-1}.
\]

If $n \equiv 1 \pmod{2}$, then
\[
q^{\alpha n} w^n G_{m,q,w}^{(\alpha)}(n) + G_{m,q,w}^{(\alpha)} = m[2]_{q^\alpha} \sum_{l=0}^{n-1} (-1)^l q^{\alpha l} w^l[l]_q^{m-1}.
\]

By (1.2), after some elementary calculations, we get
\[
[2]_{q^\alpha} t = q^\alpha \int_{\mathbb{Z}_p} t \phi(x + 1) e^{[x+1]_q t} d\mu_{-q^n}(x) + \int_{\mathbb{Z}_p} t \phi(x) e^{[x]_q t} d\mu_{-q^n}(x)
\]
\[
= \sum_{n=0}^{\infty} \left( w q^\alpha \int_{\mathbb{Z}_p} n \phi(x) [x + 1]_q^{n-1} d\mu_{-q^n}(x) + \int_{\mathbb{Z}_p} n \phi(x) [x]_q^{n-1} d\mu_{-q^n}(x) \right) \frac{t^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} \left( w q^\alpha G_{n,q,w}^{(\alpha)}(1) + G_{n,q,w}^{(\alpha)} \right) \frac{t^n}{n!}.
\]

By comparing coefficients of $\frac{t^n}{n!}$ in the above equation, we arrive at the following theorem.

Theorem 2.4 For $n \in \mathbb{Z}_+$, we have
\[
w q^\alpha G_{n,q,w}^{(\alpha)}(1) + G_{n,q,w}^{(\alpha)} = \begin{cases} [2]_{q^\alpha}, & \text{if } n = 1, \\ 0, & \text{if } n \neq 1. \end{cases}
\]

By Theorem 2.4 and (2.10), we have the following corollary.

Corollary 2.5 For $n \in \mathbb{Z}_+$, we have
\[
w q^\alpha (q G_{q,w}^{(\alpha)} + 1)^n + G_{n,q,w}^{(\alpha)} = \begin{cases} [2]_{q^\alpha}, & \text{if } n = 1, \\ 0, & \text{if } n \neq 1, \end{cases}
\]

with the usual convention of replacing $(G_{q,w}^{(\alpha)})^{n}$ by $G_{n,q,w}^{(\alpha)}$. 
Twisted $q$-Genocchi numbers and polynomials

3 Twisted $q$-Genocchi zeta function

By using twisted $q$-Genocchi numbers and polynomials with weak weight $\alpha$, twisted $q$-Genocchi zeta function with weak weight $\alpha$ and Hurwitz twisted $q$-Genocchi zeta functions with weak weight $\alpha$ are defined. These functions interpolate the twisted $q$-Genocchi numbers and twisted $q$-Genocchi polynomials with weak weight $\alpha$, respectively. From (2.4), we note that

$$
\frac{d^{k+1}}{dt^{k+1}} F_{q,w}^{(\alpha)}(t) \bigg|_{t=0} = (k + 1)[2]_{q^\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m q^{\alpha m} w^m [m]^k}{[m]^k_{q^\alpha}}
$$

$$
= G_{k+1,q,w}^{(\alpha)} (k \in \mathbb{N}).
$$

By using the above equation, we are now ready to define twisted $q$-Genocchi zeta functions.

**Definition 3.1** Let $s \in \mathbb{C}$.

$$
\zeta_{q,w}^{(\alpha)}(s) = [2]_{q^\alpha} \sum_{n=1}^{\infty} \frac{(-1)^n q^{\alpha n} w^n}{[n]_{q}^s}.
$$

(3.1)

Note that $\zeta_{q,w}^{(\alpha)}(s)$ is a meromorphic function on $\mathbb{C}$. Note that, if $q \to 1$, then $\zeta_{q,w}^{(\alpha)}(s) = \zeta(s)$ which is the Genocchi zeta functions. Relation between $\zeta_{q,w}^{(\alpha)}(s)$ and $G_{k,q,w}^{(\alpha)}$ is given by the following theorem.

**Theorem 3.2** For $k \in \mathbb{N}$, we have

$$
\zeta_{q,w}^{(\alpha)}(-k) = \frac{G_{k+1,q,w}^{(\alpha)}}{k + 1}.
$$

(3.2)

Observe that $\zeta_{q,w}^{(\alpha)}(s)$ function interpolates $G_{k,q,w}^{(\alpha)}$ numbers at non-negative integers.

By using (2.9), we note that

$$
\frac{d^{k+1}}{dt^{k+1}} F_{q,w}^{(\alpha)}(t,x) \bigg|_{t=0} = (k + 1)[2]_{q^\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m q^{\alpha m} w^m [x + m]^k_{q}}{[x + m]^k_{q}}
$$

(3.3)

$$
= G_{k+1,q,w}^{(\alpha)}(x), (k \in \mathbb{N})
$$

By (3.2) and (3.3), we are now ready to define the Hurwitz twisted $q$-Genocchi zeta functions.
Definition 3.3 Let $s \in \mathbb{C}$.
\[
\zeta_{q,w}^{(\alpha)}(s, x) = [2]_q^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n q^{\alpha n} w^n}{[n + x]^q} .
\] (3.4)

Note that $\zeta_{q,w}^{(\alpha)}(s, x)$ is a meromorphic function on $\mathbb{C}$. Observe that, if $q \to 1$ and $w = 1$, then $\zeta_{q,w}^{(\alpha)}(s, x) = \zeta(s, x)$ which is the Hurwitz Genocchi zeta functions. Relation between $\zeta_{q,w}^{(\alpha)}(s, x)$ and $G_{k,q,w}^{(\alpha)}(x)$ is given by the following theorem.

Theorem 3.4 For $k \in \mathbb{N}$, we have
\[
\zeta_{q,w}^{(\alpha)}(-k, x) = \frac{G_{k+1,q,w}^{(\alpha)}(x)}{k + 1}.
\]

Observe that $\zeta_{q,w}^{(\alpha)}(-k, x)$ function interpolates $G_{k,q,w}^{(\alpha)}(x)$ numbers at non-negative integers.

References


Received: November, 2011