Numerical Solutions of Coupled Klein-Gordon-Schrödinger Equations by Finite Element Method

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Abstract

The nonlinear coupled Klein-Gordon-Schrödinger equations describes a system of conserved scalar nucleons interacting with neutral scalar Mesons coupled with Yukawa interaction method. In this paper we derive a finite element scheme to solve these equations, we test this method for stability and accuracy, many numerical tests have been given to show the validity of the scheme.

Keywords: Klein-Gordon-Schrödinger equations, finite element method.

1 Introduction

During the past decades a wide range of physical phenomena is explained by dynamics of nonlinear waves. One of the most challenging and modern applications of the control of partial differential equations is the control of quantum mechanical system [1, 4, 5, 17-19]. In the present paper we study the standard system of nonlinear coupled Klein-Gordon-Schrödinger equations:

\[ \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 v}{\partial x^2} + 2uv = 0, \quad (x, t) \in (x_L, x_R) \times (0, T) \]  

\[ \frac{\partial^2 v}{\partial t^2} \cdot \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2}{\partial x^2} |u|^2 = 0, \quad (x, t) \in (x_L, x_R) \times (0, T) \]

where \( u(x, t) \) is a complex unknown function represent the slowly varying envelope of the highly oscillatory electric field, and \( v(x, t) \) a real scalar function denotes the fluctuation of the ion density about its equilibrium value. We supplement (1) by prescribing the initial-boundary value conditions:
and satisfy the following three conservation law

(i) The total energy conservation law

\[ e(t) = \int_{-\infty}^{\infty} (u^2 + v_t^2 + v_x^2 + |u_x|^2 - 2v|u|^2) dx \]  

(ii) The total momentum conservation law

\[ \omega(t) = \int_{-\infty}^{\infty} (\text{Im}(\bar{u}u_x) - v_t v_x) dx \]

where Im stands for the imaginary part, and \( \bar{u} \) denotes the complex conjugate of \( u \).

(iii) The charge conservation law

\[ C(t) = \int_{-\infty}^{\infty} |u|^2 dx \]

Eq. (1) consists of the classical Schrödinger equation and Klein Gordon equation. Over the last tens of years, there has been a great deal of work on the numerical solving Schrödinger equations. Recently, the multi-simplistic methods have been proposed and investigated for some important Hamiltonian partial differential equations, such as Klein Gordon Schrödinger eq. (1). Eq. (1) is a classical model described the interaction between conservative complex neutron field and neutral meson Yukawa in quantum field theory. Global strong solution in [9, 11] and stability of stationary state in [14] is proved for the Klein Gordon Schrödinger equations. Asymptotic behavior of the solutions is reported [10, 15]. The solitary wave solution of the Klein-Gordon- Schrödinger equations are obtained in [20]. Numerical studies for the KGS equations are few. We constructed an implicit scheme in [21]. However many finite difference schemes have been presented for the Zakharov equations presented [2, 4, 6, 7, 16, 21, 22].

In this paper we derive the finite elements method in section 2. The accuracy and stability of the scheme are given in section 3 and section 4. Numerical tests for single and interaction of two solitons are given in section 5. Conclusions are contained in Section 6.

2 Numerical method

For the numerical study we decompose the complex function \( u \) in (1) by

\[ u(x, t) = u_1 + i u_2, \]

also we set

\[ v(x, t) = u_3, \quad \frac{\partial v}{\partial t} = u_4 \]

Substituting in (1) we get

\[ \frac{\partial u}{\partial t} + \frac{1}{2} \lambda^{2} \frac{\partial^{2} u}{\partial x^{2}} + F(u)u = 0 \]
Numerical solutions

The finite element method [8, 12, 13] for the resulting system (8) can be described as follows

The weak form of (8) is obtained by taking $L_2$–inner product with elements of the space $H^1(x_L, x_R)$ . After integration by parts, and using the boundary conditions we obtain

$$
\left( \frac{\partial u}{\partial t}, \psi \right) - \frac{1}{2} A \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x} \right) + (G(u), \psi) = 0, \forall \psi \in H^1(x_L, x_R)
$$

where

$$
G(u) = F(u)u
$$

and $L_2$–inner product $(\ldots, \ldots)$ is interpreted pointwise and denoted by

$$
(f, g) = \int_{x_L}^{x_R} f(x)g(x)dx
$$

The interval $[x_L, x_R]$ is discretized by uniform $(N+1)$ grid points

$$
x_m = x_L + mh, \quad m = 0, 1, ..., N
$$

where the grid spacing $h$ is given by $h = (x_R - x_L)/N$. Let $S^h$ be an N dimensional subspace of $H^1$ . We choose the elements of $S^h$ to be piecewise linear hat functions and denoted by $\{\psi_m\}_{m=1}^N$. We assume the approximation of the exact $u(x, t)$ solution is given by

$$
U(x, t) = \sum_{m=0}^{N} U_m(t)\psi_m(x)
$$

where $U_m, \ m = 0, 1, ..., N$ are time dependent coefficients which are to be determined. Then the discrete form of (9) is given by

$$
\sum_{m=0}^{N} U_m(\psi_m, \psi_j) - \frac{1}{2} A \sum_{m=0}^{N} U(\psi_m(x), \psi_j) + \sum_{m=0}^{N} (G(U), \psi_j) = 0, \ j = 0, 1, ..., N
$$

where

$$
U_m = \frac{\partial u_m}{\partial t}, \quad U'_m = \frac{\partial u_m}{\partial x}
$$

In order to deal with nonlinear term in (11), we use the product approximation [8] by which

$$
(G(U), \psi_m) = \sum_{m=0}^{N} G(U_m)(\psi_m, \psi_m)
$$

Since the trail and the test function is chosen to be piecewise linear functions
\[ \psi_m(x) = \begin{cases} 
(x - x_{m-1})/h & \text{if } x_{m-1} < x < x_m \\
(x_{m+1} - x)/h & \text{if } x_m < x < x_{m+1} \\
0 & \text{otherwise} 
\end{cases} \]

We note that the solution in (11) is zero outside the interval and according to this we can write it as

\[ U(x) = \begin{cases} 
U_{m-1}(t)\psi_{m-1}(x) + U_m(t)\psi_m(x) & \text{if } x_{m-1} < x < x_m \\
U_m(t)\psi_m(x) + U_{m+1}(t)\psi_{m+1}(x) & \text{if } x_m < x < x_{m+1} \\
0 & \text{otherwise} 
\end{cases} \]

and the nonlinear term as

\[ G(U(x, t)) = \begin{cases} 
G(U_{m-1}(t))\psi_{m-1}(x) + G(U_m(t))\psi_m(x) & \text{if } x_{m-1} < x < x_m \\
G(U_m(t))\psi_m(x) + G(U_{m+1}(t))\psi_{m+1}(x) & \text{if } x_m < x < x_{m+1} \\
0 & \text{otherwise} 
\end{cases} \]

By using these definitions, the inner product in (11) can be calculated as follows

\[ \int_{x_{m-1}}^{x_{m+1}} \dot{U}_\psi m dx = \int_{x_{m-1}}^{x_m} \dot{U}_\psi m dx + \int_{x_m}^{x_{m+1}} \dot{U}_\psi m dx \]

\[ = \frac{1}{6} (\dot{U}_{m-1} + \dot{U}_m + \dot{U}_{m+1}) \]

\[ \int_{x_{m-1}}^{x_{m+1}} U_x(\psi_x)_m dx = \int_{x_{m-1}}^{x_m} U_x(\psi_x)_m dx + \int_{x_m}^{x_{m+1}} U_x(\psi_x)_m dx \]

\[ = \frac{1}{h^2} (U_{m-1} - 2U_m + U_{m+1}) \]

and

\[ \int_{x_{m-1}}^{x_{m+1}} G(U)\psi_m dx = \int_{x_{m-1}}^{x_m} G(U)\psi_m dx + \int_{x_m}^{x_{m+1}} G(U)\psi_m dx \]

\[ = \frac{1}{6} (G(U_{m-1}) + 4G(U_m) + G(U_{m+1})) \]

By making use of these integrals in (11), this will lead us to the first order ordinary differential system

\[ \left(1 + \frac{1}{6} \delta_x^2 \right) U_m + \frac{1}{2h^2} A \delta_x^2 U_m + \left(1 + \frac{1}{6} \delta_x^2 \right) G(U_m) = 0 \]  \hspace{1cm} (13)\]

where

\[ \delta_x^2 = (U_{m-1} - 2U_m + U_{m+1}) \]
2.1 Full discretization

In order to solve the system (13), we first discretize the time interval $t = nk, n = 0,1,2, \ldots$ by the grid points where $k$ is the time step size. We assume that $U_m^n$ is the approximation of the exact solution $u(x_m, t_n)$. Making use of the substitution:

$$\hat{U} = \frac{U_{m+1}^n - U_m^n}{k} \quad \text{and} \quad U_m = (U^*)_m^n$$

Where

$$(U^*)_m^n = \frac{U_{m+1}^n + U_m^n}{2}$$

which is equivalent to the implicit midpoint rule? This will lead us to the implicit nonlinear scheme:

$$(1 + \frac{1}{6} \delta_x^2) \left( \frac{U_{m+1}^n - U_m^n}{k} \right) + \frac{1}{4h^2} A \delta_x^2 (U^*)_m^n + \left( 1 + \frac{1}{12} \delta_x^2 \right) G((U^*)_m^n) = 0 \quad (14)$$

3 Accuracy of the scheme

To study the accuracy of the scheme (14) we replace the numerical solution $U_m^n$ by the exact solution $u_m^n$ in (14). Doing this the proposed scheme will be of the form

$$(1 + \frac{1}{6} \delta_x^2) \left( \frac{u_{m+1}^n - u_m^n}{k} \right) + \frac{1}{4h^2} A \delta_x^2 (u_m^n + u_{m+1}^n) + \left( 1 + \frac{1}{12} \delta_x^2 \right) G \left( \frac{u_{m+1}^n + u_m^n}{2} \right) = 0 \quad (15)$$

Taylor expansion of all terms in (15) can be displayed as follows

$$\frac{1}{k} \left( 1 + \frac{1}{6} \delta_x^2 \right) \left( u_{m+1}^n - u_m^n \right) = \frac{\partial u}{\partial t} + \frac{k}{2} \frac{\partial^2 u}{\partial t^2} + \frac{k^2}{6} \frac{\partial^2 u}{\partial t^2} + \frac{h^2}{12} \frac{\partial^3 u}{\partial t^3} + \ldots$$

$$\frac{1}{4h^2} \delta_x^2 (u_m^n + u_{m+1}^n) = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + \frac{k}{4} \frac{\partial^3 u}{\partial x^2 \partial t} + \frac{h^2}{24} \frac{\partial^4 u}{\partial x^2 \partial t^2} + \ldots$$

$$(1 + \frac{1}{6} \delta_x^2) G \left( \frac{u_{m+1}^n + u_m^n}{2} \right) = G + \frac{k}{2} \frac{\partial G}{\partial t} + \frac{h^2}{12} \frac{\partial^2 G}{\partial x^2} + \frac{h^2}{24} \frac{\partial^3 G}{\partial x^2} + \ldots$$

Substituting by this expansion in (15) we get

$$T_m^n = \left( \frac{\partial u}{\partial t} + \frac{1}{2} A \frac{\partial^2 u}{\partial x^2} + G(u) \right)_m^n + \frac{k}{2} \frac{\partial u}{\partial t} + \frac{1}{2} A \frac{\partial^2 u}{\partial x^2} + G(u) \right)_m^n$$

$$+ \frac{h^2}{12} \frac{\partial^2 u}{\partial x^2} + \frac{1}{2} A \frac{\partial^2 u}{\partial x^2} + G(u) \right)_m^n$$

$$+ \frac{k^2}{6} \frac{\partial^3 u}{\partial t \partial x^2} + \frac{h^2}{24} \frac{\partial^4 u}{\partial x^2} + \frac{h^2}{12} \frac{\partial^2 G}{\partial x^2} + \ldots$$
By using the differential eq. (8) all terms inside the brackets are equal zero, then we have

\[ T_m^n = \frac{k^2}{6} \frac{\partial^3 u}{\partial t^3} + \frac{h^2}{12} \frac{\partial^4 u}{\partial t^2 \partial x^2} + \frac{h^2}{24} \frac{\partial^4 u}{\partial t \partial x^4} + \frac{h^2}{12} \frac{\partial^2 u}{\partial x^2} + \cdots \]

Thus, \( T_m^n \) tends to zero if \( h, k \) tends to zero, we deduce that the proposed scheme is a second order in time and space, it is consistent since the local truncation error \( T_m^n \) tends to zero.

4 Stability

To study the stability of the proposed scheme, the von Neumann stability analysis will be used. This method can only be applied for linear scheme. By freezing all terms which make the scheme nonlinear [12], then eq. (8) has the form

\[ \frac{\partial u}{\partial t} + \frac{1}{2} C_1 \frac{\partial^2 u}{\partial x^2} + \alpha C_2 u = 0 \]  

(16)

where

\[ C_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/\alpha \\ -1 & -1 & 1/\alpha & 0 \end{pmatrix} \]

\( \alpha = \max \{u_1, u_2, u_3, u_4\} \)

The finite difference scheme of eq. (16) is

\[ \left( 1 + \frac{1}{6} \delta_x^2 \right) (U_m^{n+1} - U_m^n) + \frac{1}{2} r C_1 \delta_x^2 (U_m^{n+1} + U_m^n) + \frac{k \alpha}{2} C_2 \left( 1 + \frac{1}{6} \delta_x^2 \right) (U_m^{n+1} + U_m^n) = 0 \]  

(17)

where \( r = \frac{k}{2h} \). Assume that

\[ U_m^n = E^n U_0 e^{i \beta mh} \quad i = \sqrt{-1} \]  

(18)

be the test function \( \beta \in \mathbb{R} \), \( U_0 \in \mathbb{R}^4 \) and \( E \in \mathbb{R}^{4 \times 4} \) be the amplification matrix. The Von Neumann necessary condition for stability of the scheme is

\[ \max_j |\lambda_j| \leq 1, \quad j = 1, 2, 3, 4 \]

Substituting in (17) we get after manipulation the amplification matrix is given by

\[ E = (\gamma I - \omega_1 C_1 + \omega_2 C_2)^{-1} (\gamma I + \omega_1 C_1 - \omega_2 C_2) \]  

(19)

where \( \omega_1 = \frac{1}{2} r \alpha \mu, \quad \omega_2 = \frac{k}{2} \alpha \gamma, \quad \gamma = 1 - \frac{2}{3} \sin^2 \frac{\beta h}{2}, \quad \mu = -4 \sin^2 \frac{\beta h}{2} \)
The eigenvalues of the matrix $E$ are

$$\begin{bmatrix}
y - i\omega_2 & y - i\omega_2/\alpha & y + i\omega_2/\alpha \\
y + i\omega_2 & y + i\omega_2/\alpha & y - i\omega_2/\alpha \\
y - i\omega_2 & y - i\omega_2/\alpha & y + i\omega_2/\alpha \\
y + i\omega_2 & y + i\omega_2/\alpha & y - i\omega_2/\alpha
\end{bmatrix}.$$  

It is clear that all the modulus of the eigenvalues equal 1, then the scheme is unconditionally stable. the scheme is also consistent, and then according to Lax theorem the scheme is convergent.

5 Numerical results

We study the accuracy of the proposed scheme by calculating the infinity norm of the error and we experiment the conservations laws. The trapezoidal rule is used to calculate the conservation quantities and the accuracy is measured by using $L_\infty$ error norm of $u$ and $v$ which given by

$$\|Er(u)\|_\infty = \max_{0<j<N} |u(x_j, t_n) - u^n_j|, \quad \|Er(v)\|_\infty = \max_{0<j<N} |v(x_j, t_n) - v^n_j|.$$  

Two cases are considered single soliton and two soliton interactions. We take the parameters $x_L = -50.0, x_R = 50.0, h = 0.2, k = 0.01$ and the velocity $q = 0.1$

**Single soliton solution**

In this test we take

$$u(x, t) = \frac{3\sqrt{2}}{4\sqrt{1 - q^2}} \text{sech}^2 \left( \frac{1}{2\sqrt{(1-q^2)}} x \right) \exp(iqx),$$

$$v(x, t) = \frac{3}{4(1-q^2)} \text{sech}^2 \left( \frac{3}{2\sqrt{(1-q^2)}} x \right).$$

In Table 1 and Table 2, we calculate the errors and the conserved quantities. It is very clear that the method produced accurate results and keeps the conserved quantities almost constants for the charge, momentum and nearly constants for the energy. In Figure 1 shows the evolution of single soliton $u$ and $v$ at $t = 0, 1, 2, \ldots$ moving to the right.

![Figure 1: Single soliton solution left $u$ right $v$](image)
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### Table 1: Single soliton

<table>
<thead>
<tr>
<th>Time</th>
<th>$|Er(u)|_\infty$</th>
<th>$|Er(v)|_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.00538971</td>
<td>0.0539015</td>
</tr>
<tr>
<td>20</td>
<td>0.00943423</td>
<td>0.0941379</td>
</tr>
<tr>
<td>30</td>
<td>0.01168950</td>
<td>0.0116418</td>
</tr>
<tr>
<td>40</td>
<td>0.01493350</td>
<td>0.0148990</td>
</tr>
<tr>
<td>50</td>
<td>0.01904940</td>
<td>0.0189605</td>
</tr>
<tr>
<td>60</td>
<td>0.02281280</td>
<td>0.0227392</td>
</tr>
</tbody>
</table>

### Two Solitons Interaction

In this test we take the interaction of two solitons, where the initial conditions are assumed of the form

$$u(x, 0) = \sum_{j=1}^{2} \frac{3\sqrt{2}}{4(1-q_j^2)} \text{sech}^2 \left( \frac{1}{2\sqrt{(1-q_j^2)}} (x - x_j) \right) \exp \left( q_j (x - x_j) i \right),$$

$$v(x, 0) = \sum_{j=1}^{2} \frac{3}{4(1-q_j^2)} \text{sech}^2 \left( \frac{1}{2\sqrt{(1-q_j^2)}} (x - x_j) \right)$$

which represents the sum of two single solitons located at $x_1$ and $x_2$ respectively. Figure 2 display the interactions of two solitons $u$ and $v$ at $T = 0, 1, ..., 60$ with the parameters $q_1 = 0.9, q_2 = 0.9, x_1 = -30.0$ and $x_2 = 30.0$. We noticed that the two waves approaches each other interact and leave the interaction unchanged in shape and velocity.

### Table 2: Single soliton (conserved quantities)

<table>
<thead>
<tr>
<th>Time</th>
<th>Energy</th>
<th>Charge</th>
<th>Momentum</th>
</tr>
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<tbody>
<tr>
<td>10</td>
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<td>1.73641</td>
<td>0.331929</td>
</tr>
<tr>
<td>20</td>
<td>-1.184230</td>
<td>1.73641</td>
<td>0.331929</td>
</tr>
<tr>
<td>30</td>
<td>-1.184235</td>
<td>1.73641</td>
<td>0.331929</td>
</tr>
<tr>
<td>40</td>
<td>-1.184238</td>
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</tr>
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<td>50</td>
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<td>1.73641</td>
<td>0.331929</td>
</tr>
<tr>
<td>60</td>
<td>-1.184230</td>
<td>1.73641</td>
<td>0.331929</td>
</tr>
</tbody>
</table>

Figure 2: Interaction of two soliton solutions left $|u|$ right $v$
6 Conclusions

The Klein-Gordon-Schrödinger equations have important role in quantum physics. In this paper, we developed a finite element method for solving the coupled nonlinear Schrödinger equations. The resulting scheme is second order in both directions space and time, and it is unconditionally stable. The numerical experiments are presented to demonstrate the effectiveness of the scheme and keep the conservative laws. The method simulates the interaction picture clearly in the case of two solitons interaction.

References

