Robust Exponential Stabilization for Uncertain Linear Non-autonomous Control Systems with Mixed Time-varying Delays and Nonlinear Perturbations

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Abstract

This work addresses the problems of robust exponential stability and stabilization for uncertain linear non-autonomous control systems with discrete and distributed time-varying delays and nonlinear perturbations. Based on the combination of the Riccati differential equation approach and the Lyapunov-Krasovskii functional, new sufficient conditions are derived in terms of the solution of Riccati differential equations. Numerical examples illustrate the results are given.

Keywords: Robust exponential stabilization, Distributed delay, Riccati differential equation, Uncertain linear non-autonomous control delayed system

1 Introduction

In the past decades, the stability problem of uncertain linear time-delay system has attracted a lot of attention [2],[9]-[12],[14]-[16]. Because time-delay and uncertainties are frequently a source of instability or poor performances in various systems such as including physical and chemical processes, electric, biology, economic, engineering etc. The results of these works concerning Lyapunov’s direct method for time-delay system provide sufficient conditions in term of linear matrix inequalities (LMIs). It is obvious that the result of these works cannot be extended to uncertain linear non-autonomous time-delay systems due to the unsolved infinite systems of linear matrix inequalities (LMIs).
In [13], [17], [19], they shown some results for linear non-autonomous systems with time-varying delays. Moreover, many researchers have been studied the problem of stability for systems with discrete and distributed delays [6], [8], [21] such as [21] presented some stability conditions for uncertain neutral systems with discrete and distributed delays. The robust stability of uncertain linear neutral systems with discrete and distributed delays has studied in [8].

In this work, we will consider the problems of robust exponential stability and stabilization for uncertain linear non-autonomous control system with discrete and distributed time-varying delays and nonlinear perturbations. On the basis of the Lyapunov theorem and Riccati differential equation approach, new sufficient conditions for robust exponential stability and stabilization are derived in terms of the solution of Riccati differential equations, with allow us to compute simultaneously the two bounds that characterize the stability rate of the solution. Numerical examples illustrate the results will be show.

2 Problem formulation and preliminaries

We introduce some notations and definitions that will be used throughout the paper. \( \mathbb{R}^+ \) denotes the set of all real non-negative numbers; \( \mathbb{R}^n \) denotes the \( n \)-dimensional space with the vector norm \( \| \cdot \| \); \( \| x \| \) denotes the Euclidean vector norm of \( x \in \mathbb{R}^n \); \( M^{nxr} \) denotes the space of all matrices of \( (n \times r) \)-dimensions; \( A^T \) denotes the transpose of the matrix \( A \); \( A \) is symmetric if \( A = A^T \); \( I \) denotes the identity matrix; \( \lambda(A) \) denotes the set of all eigenvalues of \( A \); \( \lambda_{\text{max}}(A) = \max \{ \Re \lambda : \lambda \in \lambda(A) \} \); Matrix \( A \) is called semi-positive definite \( (A \geq 0) \) if \( x^T Ax \geq 0 \), for all \( x \in \mathbb{R}^n \); \( A \) is positive definite \( (A > 0) \) if \( x^T Ax > 0 \) for all \( x \neq 0 \); Matrix \( B \) is called semi-negative definite \( (B \leq 0) \) if \( x^T Bx \leq 0 \), for all \( x \in \mathbb{R}^n \); \( B \) is negative definite \( (B < 0) \) if \( x^T Bx < 0 \) for all \( x \neq 0 \); \( A > B \) means \( A - B > 0 \); \( A \geq B \) means \( A - B \geq 0 \); \( C([-h, 0], \mathbb{R}^n) \) denotes the space of all piecewise continuous vector functions mapping \([-h, 0] \) into \( \mathbb{R}^n \); \( BM^+(0, \infty) \) denotes the set of all symmetric semi-positive definite matrix functions bounded on \([0, \infty)\); \( * \) represents the elements below the main diagonal of a symmetric matrix.

Consider an uncertain linear non-autonomous control system with discrete and distributed time-varying delays and nonlinear perturbations of the form

\[
\begin{cases}
\dot{x}(t) = \tilde{A}(t)x(t) + \tilde{B}(t)x(t-h(t)) + f_1(t, x(t)) + f_2(t, x(t-h(t))) + \tilde{C}(t) \int_{t-r(t)}^{t} x(s)ds + D(t)u(t) \quad t \geq 0; \\
x(t_0 + \theta) = \phi(\theta), \quad \forall \theta \in [-\max\{r, h\}, 0], \\
\tilde{A}(t) = [A(t) + \Delta A(t)], \quad \tilde{B}(t) = [B(t) + \Delta B(t)], \\
\tilde{C}(t) = [C(t) + \Delta C(t)],
\end{cases}
\] (2.1)
where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^n$ is the control and $A(t), B(t), C(t), D(t) \in \mathbb{R}^{n \times n}$ are matrices function continuous in $t \geq 0$. $h(t)$ is a given time-varying discrete delay function and $r(t)$ is a given time-varying distributed delay function satisfy

$$0 \leq h(t) \leq h, \quad \dot{h}(t) \leq \delta < 1, \quad 0 \leq r(t) \leq r,$$

(2.2)

where $h, r, \delta$ are positive real numbers. The initial condition function $\phi(t) \in C([- \max \{r, h\}, 0], \mathbb{R}^n)$ denotes a continuous vector-valued initial function of $t \in [- \max \{r, h\}, 0]$ with the norm $\|\phi\| = \sup_{s \in [- \max \{r, h\}, 0]} \|\phi(s)\|$. The uncertainties $\Delta A(t), \Delta B(t)$ and $\Delta C(t)$ are time varying matrices and satisfy the condition

$$\Delta A(t) = E_1(t)F(t)M_1(t), \quad \Delta B(t) = E_2(t)F(t)M_2(t),$$

$$\Delta C(t) = E_3(t)F(t)M_3(t),$$

where $E_i(t), M_i(t), i = 1, 2, 3$ are matrices function continuous and bounded in $t \geq 0$. The uncertain matrix $F(t)$ satisfies

$$F(t)^T F(t) \leq I.$$

(2.3)

The uncertainties $f_1(.)$ and $f_2(.)$ represent the nonlinear parameter perturbations with respect to the current state $x(t)$ and the delayed state $x(t-h(t))$, respectively, and are bounded in magnitude:

$$f_1^T(t, x(t))f_1(t, x(t)) \leq \beta_1 x^T(t)x(t),$$

(2.4)

$$f_2^T(t, x(t-h(t)))f_2(t, x(t-h(t))) \leq \beta_2 x^T(t-h(t))x(t-h(t)),$$

(2.5)

where $\beta_1, \beta_2$ are given positive real numbers.

**Definition 2.1** The system (2.1) where $u(t) = 0$ is robust exponentially stable, if there exist positive real numbers $\alpha$ and $M$ such that for each $\phi(t) \in C([- \max \{r, h\}, 0], \mathbb{R}^n)$, the solution $x(t, \phi)$ of the system (2.1) where $u(t) = 0$ satisfies

$$\|x(t, \phi)\| \leq M\|\phi\|e^{-\alpha t}, \quad \forall t \in \mathbb{R}^+.$$  

If there exists the state feedback controller $u(t) = K x(t)$, where $K \in \mathbb{R}^{n \times n}$ is a constant gain matrix, the closed-loop system (2.1) is robust exponentially stable, then we say system (2.1) is robust exponentially stabilizable.

**Proposition 2.1** [18] Let $A, P, G, H, F$ be real matrices of appropriate dimensions and $P > 0, F^T F \leq I$. Then

(i) For any $\epsilon > 0 : G F H + H^T F^T G^T \leq \epsilon^{-1} G G^T + \epsilon H^T H$.

(ii) For any $\epsilon > 0$ such that $\epsilon I - H H^T > 0$,

$$(A+G F H)P(A+G F H)^T \leq A P A^T + A P H (\epsilon I - H P H^T)^{-1} H P A^T + \epsilon^{-1} G G^T.$$
Proposition 2.2 (Schur complement lemma) [21] Given constant symmetric matrices $X, Y, Z$ where $Y > 0$. Then $X + Z^TY^{-1}Z < 0$ if and only if
\[
\begin{pmatrix} X & Z^T \\ Z & -Y \end{pmatrix} < 0 \quad \text{or} \quad \begin{pmatrix} -Y & Z \\ Z^T & X \end{pmatrix} < 0.
\]

Proposition 2.3 (Cauchy Inequality) [17] For any symmetric positive definite matrix $W \in \mathbb{R}^{n \times n}$ and $x, y \in \mathbb{R}^n$, we have
\[
\pm 2x^Ty \leq x^TWx + y^TWy - 1y.
\]

3 Stability conditions

3.1 Exponential stability conditions

In this section, we first present the stability condition for the systems (2.1) without uncertainties and control by combination of the Riccati differential equation approach and the use of suitable Lyapunov-Krasovskii functional. We introduce the following notations for later use.

\[
P_{\epsilon}(t) = P(t) + \epsilon I, \quad Q(t) = 2\alpha P^2_{\epsilon}(t) + 2\alpha P_{\epsilon}(t) + \varrho I,
\]

\[
R(t) = \frac{e^{2\alpha h}}{\epsilon_1(1-\delta) - \beta_2 e^{2\alpha h}}\epsilon_2C(t)C^T(t),
\]

where $\varrho = \beta_1 + \epsilon_1 + \epsilon_2 r + \gamma$. Consider the Riccati differential equation of the form
\[
\dot{P}_{\epsilon}(t) + P_{\epsilon}(t)A(t) + A^T(t)P_{\epsilon}(t) + P_{\epsilon}(t)R(t)P_{\epsilon}(t) + Q(t) = 0. \tag{3.1}
\]

Theorem 3.1 The system (2.1) without uncertainties and control is exponentially stable if there exist positive real numbers $\alpha, \epsilon, \epsilon_1, \epsilon_2, \gamma$ such that $\epsilon_1(1-\delta) - \beta_2 e^{2\alpha h} > 0$, a matrix function $P(t) \in BM^+(0, \infty)$ and the RDE (3.1) holds. Moreover, the solution $x(t, \phi)$ satisfies the inequality
\[
\|x(t, \phi)\| \leq M\|\phi\|e^{-\alpha t}, \quad t \in \mathbb{R}^+,
\]

where
\[
M = \sqrt{\lambda_{max}P(0) + \epsilon + \epsilon_1(1-e^{-2\alpha h})} + 2\epsilon_2 r^2.
\]

Proof. Let $P_{\epsilon}(t) \in BM^+(0, \infty), t \in \mathbb{R}^+$, be a solution of the RDE (3.1). We define the following Lyapunov-Krasovskii function for system (2.1) without
uncertainties and control of the form

\[ V(t, x(t)) = x^T(t)P(t)x(t) + \epsilon_1 \int_{t-h(t)}^{t} e^{2\alpha(s-t)}x^T(s)x(s)ds + \epsilon_2 \int_{-r}^{0} \int_{t-s}^{t} e^{2\alpha(s-t)}x^T(\theta)x(\theta)d\theta ds. \]

The derivative of \( V(.) \) along the trajectories of system (2.1) without uncertainties and control is given by

\[
\dot{V}(t, x(t)) = x^T(t)\dot{P}(t)x(t) + \dot{x}(t)x(t) + x^T(t)P(t)\dot{x}(t) + x^T(t)P(t)x(t) + \epsilon_1 x^T(t)x(t) - \epsilon_1 (1-h(t))e^{-2\alpha h(t)}x^T(t-h(t))x(t-h(t)) + \epsilon_2 x^T(t)x(t) - \epsilon_2 \int_{-r}^{0} e^{2\alpha s}x^T(t+s)x(t+s)ds - 2\alpha \epsilon_1 \int_{t-h(t)}^{t} e^{2\alpha(s-t)}x^T(s)x(s)ds - 2\alpha \epsilon_2 \int_{-r}^{0} \int_{t+s}^{t} e^{2\alpha(s-t)}x^T(\theta)x(\theta)d\theta ds.
\]

\[
\leq x^T(t)\dot{P}(t)x(t) + x^T(t)A^T(t)P(t)x(t) + x^T(t)P(t)x(t) + x^T(t)P(t)x(t) + \epsilon_1 x^T(t)x(t) - \epsilon_1 (1-h(t))e^{-2\alpha h(t)}x^T(t-h(t))x(t-h(t)) + \epsilon_2 x^T(t)x(t) - \epsilon_2 e^{-2\alpha r} \int_{t-r(t)}^{t} x^T(s)x(s)ds - 2\alpha \epsilon_1 \int_{t-h(t)}^{t} e^{2\alpha(s-t)}x^T(s)x(s)ds - 2\alpha \epsilon_2 \int_{-r}^{0} \int_{t+s}^{t} e^{2\alpha(s-t)}x^T(\theta)x(\theta)d\theta ds.
\]

By using Proposition 2.3 (Cauchy Inequality), (2.4) and (2.5), we get

\[
2x^T(t)P(t)f_1(t, x(t)) \leq x^T(t)P^2(t)x(t) + f_1^T(t, x(t))f_1(t, x(t)) \leq x^T(t)P^2(t)x(t) + \beta_1 x^T(t)x(t) \tag{3.3}
\]

\[
x^T(t)P(t)f_2(t, x(t-h(t))) \leq x^T(t)P^2(t)x(t) + f_2^T(t, x(t-h(t)))f_2(t, x(t-h(t))) \leq x^T(t)P^2(t)x(t) + \beta_2 x^T(t-h(t))x(t-h(t)) \tag{3.4}
\]
From (3.2) and (3.4), we consider
\[
-\frac{[\epsilon_1(1 - \delta) - \beta_2 e^{2\alpha h}]}{e^{2\alpha h}} x^T(t - h(t)) x(t - h(t)) + x^T(t - h(t)) B^T(t) P_\epsilon(t)x(t) + x^T(t) P_\epsilon(t) B(t)x(t - h(t)) \\
= -\frac{[\epsilon_1(1 - \delta) - \beta_2 e^{2\alpha h}]}{e^{2\alpha h}} x^T(t - h(t)) - x^T(t) P_\epsilon(t) B(t) x(t) \times \\
\frac{[\epsilon_1(1 - \delta) - \beta_2 e^{2\alpha h}]}{e^{2\alpha h}} x(t - h(t)) - B^T(t) P_\epsilon(t) x(t) \\
+ \frac{[\epsilon_1(1 - \delta) - \beta_2 e^{2\alpha h}]}{e^{2\alpha h}} x^T(t) P_\epsilon(t) B(t) B^T(t) P_\epsilon(t) x(t) \\
\leq \frac{[\epsilon_1(1 - \delta) - \beta_2 e^{2\alpha h}]}{e^{2\alpha h}} x^T(t) P_\epsilon(t) B(t) B^T(t) P_\epsilon(t) x(t),
\] (3.5)
and
\[
-\epsilon_2 e^{-2\alpha r} \int_{t-r(t)}^{t} x^T(s) x(s) ds \\
+ \int_{t-r(t)}^{t} \left[ x^T(t) P_\epsilon(t) C(t) x(s) + x^T(s) C^T(t) P_\epsilon(t) x(t) \right] ds \\
\leq - \int_{t-r(t)}^{t} \left[ \epsilon_2 e^{-2\alpha r} x^T(s) - x^T(t) P_\epsilon(t) C(t) \right] e_2^{-1} e^{2\alpha r} \times \\
\left[ \epsilon_2 e^{-2\alpha r} x(s) - C^T(t) P_\epsilon(t) x(t) \right] ds \\
+ \int_{t-r(t)}^{t} \epsilon_2^{-1} e^{2\alpha r} x^T(t) P_\epsilon(t) C(t) C^T(t) P_\epsilon(t) x(t) ds \\
\leq \frac{r \epsilon_2 e^{2\alpha r}}{\epsilon_2} x^T(t) P_\epsilon(t) C(t) C^T(t) P_\epsilon(t) x(t).
\] (3.6)

From (3.2), (3.3), (3.4), (3.5) and (3.6), we obtain
\[
\dot{V}(t, x(t)) + 2\alpha V(t, x(t)) \leq x^T(t) \left[ \dot{P}(t) + A^T(t) P_\epsilon(t) + P_\epsilon(t) A(t) + 2P_\epsilon^2(t) \\
+ 2\alpha P_\epsilon(t) + \beta_1 I + \epsilon_1 I + \epsilon_2 r I + \frac{e^{2\alpha h}}{[\epsilon_1(1 - \delta) - \beta_2 e^{2\alpha h}]} P_\epsilon(t) B(t) B^T(t) P_\epsilon(t) \\
+ \frac{r \epsilon_2 e^{2\alpha r}}{\epsilon_2} P_\epsilon(t) C(t) C^T(t) P_\epsilon(t) \right] x(t).
\]

Since \( P(t) \) is the solution of (3.1). Therefore, we conclude
\[
\dot{V}(t, x(t)) + 2\alpha V(t, x(t)) \leq 0, \quad \forall t \in \mathbb{R}^+,
\]
where gives

\[ V(t, x(t)) \leq V(0, x(0))e^{-2at}, \quad \forall t \in \mathbb{R}^+. \] (3.7)

From (3.7), it is easy to see that

\[ \epsilon\|x(t)\|^2 \leq V(t, x(t)) \leq V(0, x(0))e^{-2at} \leq N\|\phi\|^2e^{-2at}, \] (3.8)

where \( N = \lambda_{\text{max}}P(0) + \epsilon + \epsilon_1 \frac{1-e^{-2ah}}{2a} + 2\epsilon r^2. \) From (3.8), we get

\[ \|x(t)\| \leq M\|\phi\|e^{-at}, \quad \forall t \in \mathbb{R}^+, \] (3.9)

where \( M = \sqrt{\frac{N}{\epsilon}}. \) This means that the system (2.1) without uncertainties and control is exponentially stable. The proof of the theorem is complete. \( \square \)

**Remark 1.** When the system (2.1) without uncertainties and control is time-invariant, using the Schur complement lemma, the RDE (3.1) can be rewritten in terms of the LMI:

\[
\begin{pmatrix}
\Delta_{11} & P_\epsilon B & P_\epsilon C & P_\epsilon \\
B^TP_\epsilon & -[\epsilon_1(1-\delta) - \beta_2e^{2ah}]e^{-2ah}I & 0 & 0 \\
C^TP_\epsilon & 0 & -\frac{\epsilon_2}{r}e^{-2ar}I & 0 \\
P_\epsilon & 0 & 0 & -\frac{1}{2}I
\end{pmatrix} < 0,
\]

where

\[ \Delta_{11} := P_\epsilon A + A^TP_\epsilon + 2\alpha P_\epsilon + \varrho I. \]

### 3.2 Robust Exponential Stability Conditions

We introduce the following notations for later use.

\[
\bar{Q}(t) = 2\alpha P_\epsilon^2(t) + 2\alpha P_\epsilon(t) + \epsilon_3M_1^T(t)M_1(t) + \varrho I, \\
\bar{R}(t) = \frac{e^{2ah}}{\epsilon_1(1-\delta) - \beta_2e^{2ah}} \left[ B(t)M_2^T(t)[\epsilon_4I - M_2(t)M_2^T(t)]^{-1}M_2(t)B^T(t) \\
+ B(t)B^T(t) + \epsilon_4^{-1}E_2(t)E_2^T(t) \right] + \frac{re^{2ar}}{\epsilon_2} \left[ C(t)C^T(t) + C(t)M_3^T(t) \right. \\
\times [\epsilon_5I - M_3(t)M_3^T(t)]^{-1}M_3(t)C^T(t) + \epsilon_5^{-1}E_3(t)E_3^T(t) \\
+ \epsilon_3^{-1}E_1(t)E_1^T(t).
\]

Consider the Riccati differential equation of the form

\[
\dot{P}_\epsilon(t) + P_\epsilon(t)A(t) + A^T(t)P_\epsilon(t) + P_\epsilon(t)\bar{R}(t)P_\epsilon(t) + \bar{Q}(t) = 0. \] (3.10)
Applying Proposition 2.1, we obtain
\[ u \]
sider Lyapunov-Krasovskii function in Theorem 3.1 for system (2.1) where
\[
\text{Proof.} \text{ We prove this theorem by similarly to prove in Theorem 3.1. We consider Lyapunov-Krasovskii function in Theorem 3.1 for system (2.1) where } u(t) = 0 \text{ and we have } \dot{A}(t) = \left[ A(t) + E_1(t)F(t)M_1(t) \right], \dot{B}(t) = \left[ B(t) + E_2(t)F(t)M_2(t) \right] \text{ and } \dot{C}(t) = \left[ C(t) + E_3(t)F(t)M_3(t) \right]. \text{ Consider}
\[
\begin{align*}
\dot{A}(t)P_1(t) + P_1(t)\dot{A}(t) &= P_1(t)A(t) + A^T(t)P_1(t) + P_1(t)E_1(t)F(t)M_1(t) \\
&\quad + M_1^T(t)F(t)E_1^T(t)P_1(t), \\
\dot{B}(t)\dot{B}^T(t) &= [B(t) + E_2(t)F(t)M_2(t)]M_2^T(t)[B(t) + E_2(t)F(t)M_2(t)]^T, \\
\dot{C}(t)\dot{C}^T(t) &= [C(t) + E_3(t)F(t)M_3(t)]M_3^T(t)[C(t) + E_3(t)F(t)M_3(t)]^T.
\end{align*}
\]
Applying Proposition 2.1, we obtain
\[
\begin{align*}
\dot{A}(t)P_1(t) + P_1(t)\dot{A}(t) &\leq P_1(t)A(t) + A^T(t)P_1(t) + \epsilon_3^4 P_1(t)E_1(t)E_1^T(t)P_1(t) \\
&\quad + \epsilon_3^4 M_1^T(t)M_1(t), \\
\dot{B}(t)\dot{B}^T(t) &\leq [B(t) + E_2(t)F(t)M_2(t)]M_2^T(t)[E_4^2(t)E_2(t)]^T, \\
\dot{C}(t)\dot{C}^T(t) &\leq [C(t) + E_3(t)F(t)M_3(t)]M_3^T(t)[E_5^2(t)E_3(t)]^T.
\end{align*}
\]
By Theorem 3.1 and (3.10), the system (2.1) where \( u(t) = 0 \) is robust exponentially stable. The proof of the theorem is complete. \qed

As an application, we consider the uncertain linear autonomous system with discrete and distributed time-varying delays of the form
\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)x(t - h(t)) + f_1(t, x(t)) + f_2(t, x(t - h(t))) \\
&\quad + C(t) \int_{t-r(t)}^t x(s) \, ds, \quad t \geq 0; \\
x(t_0 + \theta) &= \phi(\theta), \quad \forall \theta \in [-\max\{r, h\}, 0], (3.11) \\
A(t) &= \left[ A + E_1 F(t) M_1 \right], B(t) = \left[ B + E_2 F(t) M_2 \right], \\
C(t) &= \left[ C + E_3 F(t) M_3 \right].
\end{align*}
\]
where $A, B, C, E_i, M_i, i = 1, 2, 3$ are constant matrices of appropriate dimensions and the uncertainty $F(t)$ satisfies (2.3). Therefore, we obtain the result.

**Corollary 3.3** The system (3.11) is robust exponentially stable if there exist symmetric positive definite matrices $P$ and positive real numbers $\gamma, \alpha, \epsilon, \epsilon_i, i = 1, 2, ..., 5$ such that $\epsilon_1(1 - \delta) - \beta_2 e^{2\alpha h} > 0, \epsilon_4 I - M_2 M_2^T > 0, \epsilon_5 I - M_3 M_3^T > 0$ and the following LMI holds.

\[
\begin{bmatrix}
\Delta_{11} & P \epsilon B & P \epsilon C & P \epsilon B M_2^T & P \epsilon C M_3^T & P \epsilon E_1 & P \epsilon E_2 & P \epsilon E_3 & P \\
\ast & -\omega I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & -\mu I & 0 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & -\xi & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & -\kappa & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & -\omega \epsilon_3 I & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & -\omega \epsilon_4 I & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & -\mu \epsilon_5 I & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & -\frac{1}{2} I
\end{bmatrix} < 0,
\]

(3.12)

where $\Delta_{11} = P \epsilon A + A^T P \epsilon + 2 \alpha P \epsilon + \epsilon_3 M_1^T M_1 + \rho I$, $\omega = \frac{[\epsilon_1(1 - \delta) - \beta_2 e^{2\alpha h}]}{e^{2\alpha h}}$, $\mu = \frac{\epsilon}{e^{2\alpha h}}$, $\xi = \omega (\epsilon_4 I - M_2 M_2^T)$ and $\kappa = \mu (\epsilon_5 I - M_3 M_3^T)$.

**Proof.** Because (3.11) is time-invariant system with uncertainties, using Proposition 2.2 (Schur complement lemma) and Theorem 3.2. Then, the RDE (3.10) can be rewritten in terms of the LMI (3.12). The proof of the corollary is complete. $\square$

### 4 Stabilization conditions

Consider the Riccati differential equation of the form

\[
\dot{P}(t) + P(t)[A(t) + C(t)K] + [A(t) + C(t)K]^T P(t) + P(t)R(t)P(t) + \overline{Q}(t) = 0.
\]

(4.1)

**Theorem 4.1** The system (2.1) is robust exponentially stabilizable if there exist positive real numbers $\gamma, \alpha, \epsilon, \epsilon_i, i = 1, 2, ..., 5$ such that $\epsilon_1(1 - \delta) - \beta_2 e^{2\alpha h} > 0, \epsilon_4 I - M_2 M_2^T > 0, \epsilon_5 I - M_3 M_3^T > 0$, a gain matrix $K$, a matrix function $P(t) \in BM^+(0, \infty)$ and the RDE (4.1) holds. Moreover, the solution $x(t, \phi)$ satisfies the inequality

\[
\|x(t, \phi)\| \leq M\|\phi\| e^{-\alpha t}, \quad t \in \mathbb{R}^+,
\]

where

\[
M = \sqrt{\lambda_{\max} P(0) + \epsilon + \frac{\epsilon_1(1 - e^{-2\alpha h})}{2\alpha} + 2\epsilon_2 r^2}.
\]
The feedback controller of (2.1) is given by \( u(t) = Kx(t) \).

**Proof.** We prove this theorem by similarly to prove in Theorem 3.1, 3.2. The feedback controller of the system (2.1) is given by \( u(t) = Kx(t) \). □

Consider the uncertain linear autonomous control system with discrete and distributed time-varying delays of the form

\[
\begin{aligned}
\dot{x}(t) &= A(t)x(t) + B(t)x(t - h(t)) + C(t)\int_{t-h(t)}^{t} x(s)ds + f_1(t, x(t)) \\
&\quad + f_2(t, x(t - h(t))) + Du(t), & t \geq 0; \\
x(t_0 + \theta) &= \phi(\theta), & \forall \theta \in [-\max\{r, h\}, 0], \\
A(t) &= \begin{bmatrix} A + E_1 F(t) M_1 \end{bmatrix}, B(t) = \begin{bmatrix} B + E_2 F(t) M_2 \end{bmatrix}, \\
C(t) &= \begin{bmatrix} C + E_3 F(t) M_3 \end{bmatrix},
\end{aligned}
\]

where \( A, B, C, D, E_i, M_i, i = 1, 2, 3 \) are constant matrices of appropriate dimensions and the uncertainty \( F(t) \) satisfies (2.3). Therefore, we obtain the result.

**Corollary 4.2** The system (4.2) is robust exponentially stabilizable if there exist symmetric positive definite matrices \( P, K \) and positive real numbers \( \gamma, \alpha, \epsilon, \epsilon_i, i = 1, 2, ..., 5 \) such that \( \epsilon_1(1 - \delta) - \beta_2 e^{\alpha h} > 0, \epsilon_4 I - M_2 M_2^T > 0, \epsilon_5 I - M_3 M_3^T > 0 \) and the following LMI holds.

\[
\begin{bmatrix}
\Delta_{11} & P \epsilon B & P \epsilon C & P \epsilon BM_2^T & P \epsilon CM_3^T & P \epsilon E_1 & P \epsilon E_2 & P \epsilon E_3 & P \epsilon \\
* & -\omega I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & -\mu I & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -\xi & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -\kappa & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -\omega \epsilon_3 I & 0 & 0 & 0 \\
* & * & * & * & * & * & -\omega \epsilon_4 I & 0 & 0 \\
* & * & * & * & * & * & * & -\mu \epsilon_5 I & 0 \\
* & * & * & * & * & * & * & * & -\frac{1}{2} I
\end{bmatrix} < 0,
\]

where \( \Delta_{11} = P \epsilon [A + DK] + [A + DK]^T P \epsilon + 2\alpha P \epsilon + \epsilon_3 M_1^T M_1 + \varrho I, \omega = \epsilon_1(1 - \delta) - \beta_2 e^{\alpha h}, \mu = \frac{\epsilon_2}{e^{2\alpha h}}, \xi = \omega(\epsilon_4 I - M_2 M_2^T) \) and \( \kappa = \mu(\epsilon_5 I - M_3 M_3^T) \).

The feedback controller of (4.2) is given by \( u(t) = Kx(t) \).

**5 Numerical examples**

In order to illustrate the effectiveness of our result presented in Section 3 and 4, we consider examples.
Example 5.1 Consider the linear non-autonomous system with discrete and distributed time-varying delays and nonlinear perturbations of the form
\[
\begin{cases}
\dot{x}(t) = A(t)x(t) + B(t)x(t - h(t)) + f_1(t, x(t)) + f_2(t, x(t - h(t))) \\
+ C(t) \int_{t-r(t)}^{t} x(s)ds \\
x(t_0 + \theta) = \phi(\theta),
\end{cases}
\] with any initial function \(\phi(t)\) and time-delay function \(h(t) = \frac{1}{2} \sin^2(t)\) and \(r(t) = \frac{1}{2} \cos^2(t)\)

\[
A(t) = \begin{bmatrix}
-2 & -e^{2t} - 1 \\
0 & -e^{-2t} - 2
\end{bmatrix},
\]
\[
B(t) = \begin{bmatrix}
\sqrt{(2-\sqrt{2})(1+4e^{-2t} - e^4)} & 0 \\
4\sqrt{2}(e^{-2t} + e^4) & \sqrt{(2-\sqrt{2})(e^{2t} + 4e^{-2t} - 1)}
\end{bmatrix},
\]
\[
C(t) = \begin{bmatrix}
\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\
\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}}
\end{bmatrix}, \quad f_1(t, x(t)) = \begin{bmatrix}
\frac{x_2(t) \sin t}{2} \\
\frac{x_3(t) \cos t}{2\sqrt{2}}
\end{bmatrix},
\]

\[
f_2(t, x(t - h(t))) = \begin{bmatrix}
\frac{x_1(t-h(t)) \cos t}{2} \\
\frac{x_2(t-h(t)) \sin t}{4}
\end{bmatrix},
\]

It is easy to see that \(h = \frac{1}{2}, \ r = \frac{1}{2}, \ \beta_1 = \frac{1}{8}, \ \beta_2 = \frac{1}{16}, \ \text{and} \ \dot{h}(t) = \frac{1}{2} \sin(2t)\) and then \(\delta = \frac{1}{2}\). Taking \(\epsilon = 1, \ \alpha = \gamma = \frac{1}{2}\) and \(\epsilon_1 = \epsilon_2 = \frac{1}{4}\), we have

\[
\epsilon_1(1-\delta) - \beta_2 e^{2\alpha h} = \frac{1}{8} - \frac{\sqrt{e}}{16} > 0.
\]

It can verify that the matrix \(P(t) = \begin{bmatrix} e^{-2t} & 0 \\
0 & 1 \end{bmatrix}\) is a solution of RDE (3.1). By Theorem 3.1, the system (5.1) is exponentially stable and the solution satisfies

\[
\|x(t, \phi)\| \leq \sqrt{\frac{19}{8} - \frac{1}{2\sqrt{e}}} \|\phi\|e^{-\frac{t}{2}}, \quad t \in \mathbb{R}^+.
\]

Example 5.2 Consider the uncertain linear non-autonomous control system with discrete and distributed time-varying delays and nonlinear perturbations (2.1) with any initial function \(\phi(t)\) and time-delay function \(h(t) = \frac{1}{2} \sin^2(t)\)
and \( r(t) = \frac{1}{2} \cos^2(t) \)

\[
A(t) = \begin{bmatrix} e^{4t} & 0 \\ 0 & 1 \end{bmatrix}, \quad B(t) = \begin{bmatrix} (\sqrt{2-\sqrt{3}})e^{-2t} \\ 8e^{\frac{3}{2}} \sqrt{(2-\sqrt{3})t} \end{bmatrix}, \\
C(t) = \begin{bmatrix} \frac{e^{-2t}}{4\sqrt{2e^t}} & 0 \\ 0 & e^{-2t} \end{bmatrix}, \quad D(t) = \begin{bmatrix} 2 + \frac{e^{4t}}{6} + \frac{e^{-4t}}{3} \\ 0 \\ 3 \end{bmatrix}, \\
f_1(t, x(t)) = \begin{bmatrix} x_1(t) \sin t \\ x_2(t) \cos t \end{bmatrix}, \quad f_2(t, x(t) - h(t))) = \begin{bmatrix} x_2(t-h(t)) \cos t \\ x_1(t-h(t)) \sin t \end{bmatrix}, \\
E_1(t) = \begin{bmatrix} \frac{e^{4t}}{2} & 0 \\ 0 & e^{-2t} \end{bmatrix}, \quad M_1(t) = \begin{bmatrix} e^{2t} & e^{-2t} \\ e^{-2t} & e^{2t} \end{bmatrix}, \\
E_2(t) = \begin{bmatrix} \frac{e^{-2t}}{2\sqrt{2e^t}} & 0 \\ 0 & e^{-4t} \end{bmatrix}, \quad M_2(t) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \\
E_3(t) = \begin{bmatrix} \frac{e^{-2t}}{4\sqrt{2e^t}} & 0 \\ 0 & e^{-4t} \end{bmatrix}, \quad M_3(t) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix},
\]

It is easy to see that \( h = \frac{1}{2}, r = \frac{1}{2}, \beta_1 = \frac{1}{8}, \beta_2 = \frac{1}{16} \) and \( \dot{h}(t) = \frac{1}{2} \sin(2t) \) and then \( \delta = \frac{1}{2} \). Taking \( \epsilon = \epsilon_3 = 1, \alpha = \gamma = \epsilon_4 = \epsilon_5 = \frac{1}{2} \) and \( \epsilon_1 = \epsilon_2 = \frac{1}{4} \), we have

\[
\epsilon_1(1 - \delta) - \beta_2 \epsilon^{2\alpha h} = \frac{1}{8} - \sqrt{\frac{e}{16}} > 0, \quad \epsilon_4 I - M_2(t) M_2^T(t) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} > 0,
\]

\[
\epsilon_5 I - M_3(t) M_3^T(t) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} > 0.
\]

We can verify that the matrix \( P(t) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-4t} \end{bmatrix} \) is a solution of the RDE (4.1).

By Theorem 4.1, the system (2.1) is robustly exponentially stabilizable and the feedback stabilizing control is given by

\[
u(t) = K x(t) = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} x(t), \quad t \geq 0.
\]

The solution of system (2.1) satisfies

\[
\|x(t, \phi)\| \leq \sqrt{\frac{19}{8} - \frac{1}{2\sqrt{e}} \|\phi\| e^{-\frac{t}{2}}}, \quad t \in \mathbb{R}^+.
\]

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References


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