Estimation of the Parameters in an Alternating Repair Model Using a Pareto Intensity of the Second Kind

Soufiane Gasmi
École Superieure des Sciences et Techniques de Tunis
BP 56, 1008 Bab Menara, University of Tunis, Tunisia
soufiane.kasmi3@gmail.com

Abstract

The Pareto intensity of the second kind is often used in survival analysis of technical products and in the modeling of extreme events. This intensity is especially useful as a failure model analyzing the reliability of different types of systems. Unlike exponentially decreasing functions, the Pareto intensity is a slowly decreasing function. In this paper we develop statistical methods for an alternating repair model using the Pareto intensity of the second kind. The maximum likelihood estimator is considered for determining the estimations of the model parameters. The distribution of the life times after perfect repairs and imperfect repairs are obtained. The estimation of the Fisher information matrix is given. Simultaneous confidence regions based on the likelihood ratio statistics are developed for the estimators of the parameters. Finally simulation study will be given.

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1 Introduction

The purpose of the paper is to present statistical methods and results of an alternating repair model. The most commonly used models for the failure process of a repairable system are known as minimal repair or “as bad as old” and perfect repair or ”as good as new”. In the minimal repair case, each repair leaves the system in the same state as it was before failure, we obtain a
nonhomogeneous Poisson processes (NHPP). In the perfect repair case, each repair is perfect and leaves the system as if it were new, we obtain a renewal processes (RP). It is well known in practice that the reality is between these two extreme cases. The repair may not yield a functioning item which is as good as new and the minimal repair assumption seems to be to pessimistic in repair strategies. From this it is seen that the imperfect repair is of great significance in practice. In this paper we study a related class of models, called the alternating repair model (ARM) and therefore clearly include the RP model as special case. The repair effect in this model is expressed by a reduction of the system virtual age. Some authors have proposed more general models for repairable systems. A very important class of general models using the virtual age process is introduced by Kijima (1989) [15] and Stadje and Zuckerman (1991) [21].

One of the most famous models in reliability is the model of Brown and Proschan (1983) [7] which assumed that repair is perfect with probability \( p \) and minimal with probability \( (1 - p) \). Block et al. (1993) [5] generalized this model to the case where \( p \) is time-dependent. Stadje and Zuckerman (1991) discussed optimal maintenance policies under general repair. Each of these models includes the RP and the NHPP as special cases.

Several models concerning the stochastic behavior of repairable systems are insufficiently flexible to allow for an imperfect repair, this can make the state of the system immediately after repair intermediate between that of a new system and that obtained by minimal repair. The models I and II of Kijima are modified by Baxter, Kijima and Tortorella (1996) [4]. The new developed model is more flexible than its predecessors and it is defined in terms of two distribution functions that admit natural realistic interpretations.


The Pareto intensity of the second kind, known as Pearson’s Type VI intensity (Johnson et al. (1994)) [14] has been found to provide a good model in biomedical problems, such as survival time following a heart transplant (Bain and Engelhardt (1992)) [2] or to model incomes. This intensity is used by Dyer (1981) [9] to analyse the annual wage data of production line workers in a large industrial firm. The Pareto distribution of the second kind has a decreasing failure intensity and is often used in survival analysis of technical products and in the modeling of extreme events.

The paper is organized in the following way. In Section 2 we introduce the alternating repair model based on the general model of repairable systems
proposed by Last and Szekli (1998) [17]. In this model, repairs affect the failure intensity at any instant via a virtual age process [15]. In Section 3 we introduce the maximum likelihood estimators of the scale and shape parameters of the Pareto intensity of the second kind. In Section 4 we develop the statements for the distribution of the operating time between two failures. The knowledge of these distributions is of great significance and is needed for the modeling and simulation of failure repair processes. In Section 5 estimation of the Fisher information matrix and asymptotic confidence bounds are given. Simultaneous confidence regions based on the likelihood ratio statistics are developed in Section 6. The results obtained are applied on sets of simulated data in Section 7. Finally, Section 8 provides a conclusion.

2 The model

We consider a general model of repairable systems introduced by Last and Szekli (1998) [17]. The system starts working with a prescribed failure rate $\lambda_1(t) = \lambda(t)$. Let $t_1$ denote the random time, where the system falls out. At this time $t_1$ the system will be repaired with the random degree $z_1$. The degree of repair is between 0 and 1, where the case of 0 corresponds to the minimal repair and the case of 1 to the perfect repair. The age of the system is decreased to $(1 - z_1)t_1$ which is called the virtual age of the system at time $t_1$ and is denoted by $V_1$. The distribution of the time until the next failure has then the failure rate $\lambda_2(t) := \lambda(t-t_1+V_1)$. Assume now that $t_k$ is the time of the $k$-th ($k \geq 1$) failure and that $z_k$ is the degree of repair at that time. After repair the failure rate of the $(k+1)$-th waiting time until the next failure is determined by:

$$\lambda_{k+1}(t) := \lambda(t - t_k + V_k), \quad t \geq 0, \ k \geq 0,$$

where $V_k := (1 - z_k)(V_{k-1} + t_k - t_{k-1}), \quad V_0 := 0, \ t_0 := 0$.

**Definition 2.1** *The process defined by $V(t) := t - t_k + V_k$, $t_k \leq t < t_{k+1}$, $k \geq 1$, is called the virtual age process.*

This model includes the special cases of classical perfect repair, minimal repair, imperfect repair and general repairable systems of Kijima (1989) [15], Brown and Proschan (1983) [7], Stadje and Zuckerman (1991) [21].

In this paper we introduce now the alternating repair model, which is a special case from the general repair model. It is assumed that:

**Assumption 1**
All repair times are small and can be neglected.

**Assumption 2**
The failure intensity of the system follows a Pareto intensity of the second kind
Estimation of parameters

(Howlader and Hossain (2002)) [13]:

$$\lambda(x, \theta) = \frac{\beta}{\alpha} \left( 1 + \frac{x}{\alpha} \right)^{-1}, \beta > 0, \alpha > 0, \text{ where } \theta = (\alpha, \beta).$$

**Assumption 3**

Repairs affect the failure intensity at any instant via a virtual age process.

**Assumption 4**

When the system fails, one of the two types of identifiable repair actions are possible, an imperfect repair with degree $c$ or a perfect repair.

**Assumption 5**

After failure we have an alternating sequence of imperfect repair with degree $c$ and perfect repair.

Let $(t_k)_{k=1,2,...}$ be the sequence of failure times and $(z_k')_{k=1,2,...}$ denotes the sequence of degrees of repair.

Let $N(t) = \sum_{k=1}^{\infty} 1(t_k \leq t)$ be the number of failures until $t$, where $1(A)$ is the indicator function of the event $A$.

Under the assumption 2, we suppose that the failure intensity exists and has the Pareto form of the second kind.

If we put $z_k = 1 - z_k'$ in the general repair model, where $z_k'$ takes values 0 for perfect repair and $c$ ($0 < c < 1$) for imperfect repair and we assume that the virtual age process is given by:

$$V(t) = \begin{cases} 
    t - t_{2k-2} & \text{for } t_{2k-2} \leq t < t_{2k-1} \\
    t - [(1 - c)t_{2k-1} + ct_{2k-2}] & \text{for } t_{2k-1} \leq t < t_{2k} 
\end{cases} \quad (k = 1, 2, \ldots),$$

then we obtain the alternating repair model.

**Remark 1**

1. If the degree of repair $c$ is equal 0, only perfect repairs appear in the model, and we obtain a RP.

2. If the degree of repair $c$ is equal 1, we obtain then an alternating repair model with the alternation between minimal repair and perfect repair.

3. If $0 < c < 1$ we obtain then an alternating repair model with the alternating sequence of imperfect repair with degree $c$ and perfect repair.

We consider now a marked point process $\Phi = ((t_k, z_k')) ([17])$. $\Phi$ is described by the counting process $\{N(t), t \geq 0\}$ with intensity $\lambda(V(t), \theta)$ where $V(t) := t - t_k + V_k, \quad t_k \leq t < t_{k+1}, \; k = 1, 2, \ldots$ and by marks $z_k'$ describing the degree of repair at time point $t_k$.

A realization of the virtual age process is shown in figure 1. Here $t_i$ ($i = 1, \ldots, 5$) are failure times, at times $t_1$ and $t_3$ we have an imperfect repair and at times $t_2$ and $t_4$ we have a perfect repair.
At the following we assume that the degree of repair \( c \) satisfied \( 0 < c < 1 \).

Let \( N_1(t) = \sum_{k=1}^{\infty} \mathbf{1}\{t_k \leq t\} \mathbf{1}\{z'_k = c\} \) be the number of imperfect repair until \( t \) and \( N_2(t) = \sum_{k=1}^{\infty} \mathbf{1}\{t_k \leq t\} \mathbf{1}\{z'_k = 0\} \) be the number of perfect repair until \( t \).

It is easy to see, that \( N(t) = N_1(t) + N_2(t) \). In the following we assume that \( K(t) = N_1(t) - N_2(t) \), where \( K(t) \) determined the type of the last repair. If \( K(t) = 0 \), then the last repair is a perfect repair and if \( K(t) = 1 \), then the last repair is an imperfect repair.

\[
\ln L(t; \theta) = \sum_{k=1}^{N(t)} \ln \lambda(V(t_k^-)) - \frac{t}{0} \lambda(V(s))ds, \tag{1}
\]

where \( V(t_k^-) = t_k - t_{k-1} + V_{k-1} \) and \( V_k, k = 1, \ldots, N(t) \) depend on the degree of repair which must be observable. The first term contains all failures and the second term contains the information about the working periods without failures. We obtain then:

\[
\ln L(t; \theta) = \sum_{k=1}^{N_1(t)} \ln \lambda(V(t_{2k-1}^-)) + \sum_{k=1}^{N_2(t)} \ln \lambda(V(t_{2k-1}^-)) - \int_{0}^{t} \lambda(V(s))ds. \tag{2}
\]

It follows:

\[
\ln L(t; \theta) = N(t) \ln \beta - \beta A(t, \alpha) - \sum_{k=1}^{N_1(t)} \ln(\alpha + t_{2k-1} - t_{2k-2})
\]

**Figure 1:** Virtual age process

### 3 Maximum likelihood estimators

Based on the above observations and such a virtual age process, it is straightforward to develop the log likelihood function for this model. The resultant LL function for observation of point process is of the form (Andersen et al. (1992)) [1] and (Last and Brandt (1995)) [16]:
Estimation of parameters

\[ - \sum_{k=1}^{N_2(t)} \ln(\alpha + t_{2k} - (1-c)t_{2k-1} - ct_{2k-2}), \]

(3)

where

\[ A(t, \alpha) = \sum_{k=1}^{N_1(t)} \ln \left( 1 + \frac{t_{2k-1} - t_{2k-2}}{\alpha} \right) + \sum_{k=1}^{N_2(t)} \ln \left( 1 + \frac{t_{2k} - t_{2k-1}}{\alpha + c(t_{2k-1} - t_{2k-2})} \right) + \]

\[ 1\{K(t) = 1\} \left\{ \ln \left( 1 + \frac{t - t_{N(t)}}{\alpha + c(t_{N(t)} - t_{N(t)-1})} \right) \right\} + \]

\[ 1\{K(t) = 0\} \left\{ \ln \left( 1 + \frac{t - t_{N(t)}}{\alpha} \right) \right\}. \]

The MLE of the parameters \(\alpha\) and \(\beta\) can be obtained by solving the following system of likelihood equations given by:

\[ \frac{\partial \ln L(t; \theta)}{\partial \alpha} = 0 \]

(4)

\[ \frac{\partial \ln L(t; \theta)}{\partial \beta} = 0. \]

(5)

It is possible to determine the shape parameter \(\beta\) explicitly from solving the likelihood equation (5). We obtain then:

\[ \hat{\beta} = \frac{N(t)}{A(t, \hat{\alpha})}. \]

(6)

\(\beta\) involves the parameter estimation in terms of Pareto intensities. Here, this estimator depends on the virtual age of the system and the number of failures \(N(t)\). Using the likelihood equation (4), the point estimator of the scale parameter \(\alpha\) can be found by numerical solve of the following equation:

\[ M(t, \hat{\alpha}) \left( 1 + \frac{A(t, \hat{\alpha})}{N(t)} \right) - \left( N_1(t)\hat{\alpha}^{-1} + \sum_{k=1}^{N_2(t)} (\hat{\alpha} + c(t_{2k-1} - t_{2k-2}))^{-1} \right) + \]

\[ 1\{K(t) = 1\} \left\{ (\hat{\alpha} + t - (1-c)t_{N(t)} - ct_{N(t)-1})^{-1} - (\hat{\alpha} + c(t_{N(t)} - t_{N(t)-1}))^{-1} \right\} \]

\[ + 1\{K(t) = 0\} \left\{ (\hat{\alpha} + t - t_{N(t)})^{-1} - \hat{\alpha}^{-1} \right\} = 0. \]

(7)

Where

\[ M(t, \hat{\alpha}) = \sum_{k=1}^{N_1(t)} (\hat{\alpha} + t_{2k-1} - t_{2k-2})^{-1} + \sum_{k=1}^{N_2(t)} (\hat{\alpha} + t_{2k} - (1-c)t_{2k-1} - ct_{2k-2})^{-1}. \]
4 Statements for the distribution of the operating time between two failures

Let $X_k$ be the $k$-th operating time between two failures. We have then $X_k = t_k - t_{k-1}$, $k = 1, 2, \ldots$. The knowledge of the distribution of the $k$-th operating time between two failures is of great importance and is needed for the simulation of this alternating repair process.

**Assumption 6**

We consider the distribution of $X_k$ and we suppose that the failure repair process is realized up to $t = t_k - t_{k-1}$, and all failure times used before are also known.

**Theorem 4.1** Let $X_k = t_k - t_{k-1}$ ($k = 1, 2, \ldots$) be the $k$-th operating time between two failures and let $t^*$ ($t^* \geq t_{k-1}$) be fixed. Hence we obtain:

$$P(X_k \geq u \mid X_k \geq t^* - t_{k-1}) = \left( \frac{\alpha + t^* - [(1 - z_k')t_{k-1} + z_k't_{k-2}]}{\alpha + u + z_k'(t_{k-1} - t_{k-2})} \right)^\beta, \quad (8)$$

where $u \geq t^* - t_{k-1}$.

**Proof.**

According to Brémaud (1981) [6] we obtain:

$$P(X_k \geq u) = \exp \left\{ - \int_{t_{k-1}}^{t_{k-1}+u} \lambda(x, \theta) dx \right\}. \quad (9)$$

Because of

$$P(X_k \geq u \mid X_k \geq t^* - t_{k-1}) = \frac{P(X_k \geq u)}{P(X_k \geq t^* - t_{k-1})}$$

the equation (8) follows immediately.

**Remark 2**

Under the assumption 6, if we put $t^* = t_{k-1}$ in equation (8) it follows that

$$P(X_k \geq u) = \left( \frac{\alpha + z_k'(t_{k-1} - t_{k-2})}{\alpha + u + z_k'(t_{k-1} - t_{k-2})} \right)^\beta. \quad (10)$$

**Case 1:** If $z_k' = 0$, $k = 1, 2, \ldots$, then the life times $X_k$ after perfect repair follows a Pareto distribution function of the second kind:

$$F_{X_k}(u) = \begin{cases} 0 & \text{for } u \leq 0, \\ 1 - \left( \frac{\alpha}{\alpha + u} \right)^\beta & \text{for } u > 0. \end{cases}$$
Case 2: If \( z'_k \in (0, 1), k = 1, 2, \ldots \), then the life times \( X_k \) after imperfect repair follows a truncated Pareto distribution function of the second kind:
\[
F_{X_k}(u) = \begin{cases} 
0 & \text{for } u \leq 0, \\
1 - \left( \frac{\alpha + z'_k(t_{k-1} - t_{k-2})}{\alpha + z'_k(t_{k-1} - t_{k-2}) + u} \right)^\beta & \text{for } u > 0.
\end{cases}
\]

5 Asymptotic confidence bounds

In this section we have to derive the approximate confidence intervals of the parameters \( \alpha \) and \( \beta \) based on the asymptotic distributions of the MLE of the unknown parameters. Because the MLE of the vector \( \theta = (\alpha, \beta) \) is not obtained in closed form, it is not possible to derive the exact distribution of the MLE. In this section, we derive the Fisher information matrix needed to define confidence regions of the parameters.

Let \( I(\theta) \) be the Fisher information matrix of the vector of unknown parameters \( \theta = (\theta_1, \theta_2) \). Let \( \theta_1 = \alpha \) and \( \theta_2 = \beta \). The elements of the 2x2 matrix \( I(\theta), I_{r,s}(\theta), r, s = 1, 2, \) can be approximated by \( I_{r,s}(\hat{\theta}) = -\frac{\partial^2 \ln L(t; \theta)}{\partial \theta_r \partial \theta_s} \) evaluated at \( \theta = \hat{\theta} \) (Gasmi (2011)) [10].

In this case \( n \) independent failure repair processes are observed. Let \( l \in \{1, 2, \ldots, n\} \), now we define the following notations:
- \( N_l(t) \), the number of failures until \( t \) for the \( l \)-th failure repair process.
- \( N_{l1}(t) \), the number of perfect repair until \( t \) for the \( l \)-th failure repair process.
- \( N_{l2}(t) \), the number of imperfect repair until \( t \) for the \( l \)-th failure repair process.
- \( t_{l1}, \ldots, t_{lN_l(t)} \), failure times of the \( l \)-th failure repair process.
- \( x_{l1}, \ldots, x_{lN_l(t)} \), operating times of the \( l \)-th failure repair process.
- \( L_l(t; \theta) \), the likelihood function of the \( l \)-th failure repair process.

Definition 5.1 Let \( k = 1, 2, \ldots, N_l(t) \).

1. \( x_{l,k} = t_{l,k} - t_{l,k-1} \) is the operating time between two successive failures of the \( l \)-th failure repair process.
2. \( x_{l,2k} \) is the operating time after imperfect repairs of the \( l \)-th failure repair process.
3. \( x_{l,2k-1} \) is the operating time after perfect repairs of the \( l \)-th failure repair process.

The following theorem provides a basis for the approximate confidence intervals of the parameters \( \alpha \) and \( \beta \).
Theorem 5.2 If we observe \( n \) independent failure repair processes and under the above given assumptions, the following approximations of the Fisher information are then obtained:

\[
I_{1,1}(\hat{\theta}) = \sum_{t=1}^{n} B(x_t, \hat{\theta}),
\]

\[
I_{1,2}(\hat{\theta}) = -\frac{1}{\beta} \sum_{t=1}^{n} C(x_t, \hat{\alpha}),
\]

\[
I_{2,2}(\hat{\theta}) = \frac{1}{\beta^2} \sum_{t=1}^{n} N^t(t).
\]

Where

\[
B(x_t, \hat{\theta}) = \hat{\beta} \left[ \sum_{k=1}^{N_1^t(t)} \left\{ \frac{1}{\hat{\alpha}^2} - \frac{1}{(\hat{\alpha} + x_{t,2k-1})^2} \right\} + \sum_{k=1}^{N_2^t(t)} \left\{ \frac{1}{(\hat{\alpha} + cx_{t,2k-1})^2} \right\} - \frac{1}{(\hat{\alpha} + x_{t,2k} + cx_{t,2k-1})^2} \right] + 1\{K(t) = 0\} \left\{ \frac{1}{\hat{\alpha}^2} - \frac{1}{(\hat{\alpha} + t - t_{i,N^t(t)})^2} \right\} + \right.
\]

\[
1\{K(t) = 1\} \left\{ \frac{1}{(\hat{\alpha} + cx_{t,N^t(t)})^2} - \frac{1}{(\hat{\alpha} + cx_{t,N^t(t)} + t - t_{i,N^t(t)})^2} \right\}
\]

\[
- \left[ \sum_{k=1}^{N_1^t(t)} \frac{1}{(\hat{\alpha} + x_{t,2k-1})^2} + \sum_{k=1}^{N_2^t(t)} \frac{1}{(\hat{\alpha} + x_{t,2k} + cx_{t,2k-1})^2} \right],
\]

and

\[
C(x_t, \hat{\alpha}) = \sum_{k=1}^{N_1^t(t)} \frac{1}{(\hat{\alpha} + x_{t,2k-1})} + \sum_{k=1}^{N_2^t(t)} \frac{1}{(\hat{\alpha} + x_{t,2k} + cx_{t,2k-1})}.
\]

The observed information matrix \( I \) for this model is

\[
I = \begin{pmatrix}
I_{1,1}(\theta) & I_{1,2}(\theta) \\
I_{2,1}(\theta) & I_{2,2}(\theta)
\end{pmatrix}.
\]

As next the variance-covariance matrix \( V \) is the inversion of the observed information matrix \( I \).

\[
V = \begin{pmatrix}
V_{1,1}(\theta) & V_{1,2}(\theta) \\
V_{2,1}(\theta) & V_{2,2}(\theta)
\end{pmatrix} = I^{-1}.
\]

It follows then that the asymptotic distribution of the MLE \((\hat{\alpha}, \hat{\beta})\) given by Miller (1981) [19]:

\[
\begin{pmatrix}
\hat{\alpha} \\
\hat{\beta}
\end{pmatrix} \sim N\left( \begin{pmatrix}
\alpha \\
\beta
\end{pmatrix}, \begin{pmatrix}
V_{1,1}(\theta) & V_{1,2}(\theta) \\
V_{2,1}(\theta) & V_{2,2}(\theta)
\end{pmatrix} \right).
\]

(11)
Estimation of parameters

If we replace the parameters \( \alpha, \beta \) by the corresponding MLE’s, we get then an estimate of the variance-covariance matrix \( V \), denoted by \( \hat{V} \) and defined as follows:

\[
\hat{V} = \begin{pmatrix}
V_{1,1}(\hat{\theta}) & V_{1,2}(\hat{\theta}) \\
V_{2,1}(\hat{\theta}) & V_{2,2}(\hat{\theta})
\end{pmatrix} = \begin{pmatrix}
I_{1,1}(\hat{\theta}) & I_{1,2}(\hat{\theta}) \\
I_{2,1}(\hat{\theta}) & I_{2,2}(\hat{\theta})
\end{pmatrix}^{-1}.
\] (12)

By using the relationship (11), we approximate the 100(1 - \( \nu \))\% confidence intervals for the parameters \( \alpha, \beta \) respectively as:

\[
\hat{\alpha} \pm z_{\frac{\nu}{2}} \sqrt{V_{1,1}(\hat{\theta})} \quad \text{and} \quad \hat{\beta} \pm z_{\frac{\nu}{2}} \sqrt{V_{2,2}(\hat{\theta})},
\] (13)

where \( z_{\frac{\nu}{2}} \) is the upper \( \frac{\nu}{2} \)-th percentile of the standard normal distribution.

6 Simultaneous confidence regions based on the Likelihood ratio

It is known that under regularity conditions (Peers 1971) [20] the log likelihood ratio \( q = 2\{\ln L(t; \hat{\theta}) - \ln L(t; \theta)\} \) converges in distribution to a central \( \chi^2 \)-distribution with 2 degrees of freedom, where \( \hat{\theta} = (\hat{\alpha}, \hat{\beta}) \) the maximum likelihood estimator of the parameters and \( \theta = (\alpha, \beta) \) the unknown parameter of interest.

The simultaneous confidence regions is defined by the inequality \( q \leq \chi^2_{1-\gamma,2} \), where \( \chi^2_{1-\gamma,2} = -2\ln \gamma \) is the \( (1 - \gamma) \)-quantil of the \( \chi^2 \)-distribution with 2 degrees of freedom.

The log likelihood function of the \( l \)-th failure repair process is defined as:

\[
\ln L_l(t; \theta) = N_l^1(t) \ln \beta - \sum_{k=1}^{N_l^1(t)} \ln (\alpha + x_{l,2k-1}) - \sum_{k=1}^{N_l^2(t)} \ln (\alpha + x_{l,2k} + cx_{l,2k-1})
\]

\[
- \beta \left\{ \ln \left( 1 + \frac{x_{l,2k-1}}{\alpha} \right) \right\} + \sum_{k=1}^{N_l^2(t)} \left\{ \ln \left( 1 + \frac{x_{l,2k}}{\alpha + cx_{l,2k-1}} \right) \right\} + 1\{K(t) = 0\}
\]

\[
\left\{ \ln \left( 1 + \frac{t - t_{l,N_l^1(t)}}{\alpha} \right) \right\} + 1\{K(t) = 1\} \left\{ \ln \left( 1 + \frac{t - t_{l,N_l^1(t)}}{\alpha + cx_{l,N_l^1(t)}} \right) \right\}.
\]

Using the likelihood equation:

\[
\frac{\partial \ln L_l(t; \theta)}{\partial \alpha} \bigg|_{\theta = \hat{\theta}} = 0.
\]
The border of the simultaneous confidence regions using the likelihood ratio is given as follows:

\[ 2\{\ln L_t(t; \hat{\theta}) - \ln L_t(t; \theta)\} = -2\ln \gamma. \]  \hspace{1cm} (14)

We obtain then the following simultaneous confidence regions for \( n \) independent failure repair processes:

\[
\sum_{l=1}^{n} \left\{ \sum_{k=1}^{N_1(t)} \ln \left( \frac{\alpha + x_{l,2k-1}}{\alpha + x_{l,2k-1}} \right) + \sum_{k=1}^{N_2(t)} \ln \left( \frac{\alpha + x_{l,2k} + cx_{l,2k-1}}{\alpha + x_{l,2k} + cx_{l,2k-1}} \right) \right\} + \beta \sum_{l=1}^{n} D^l(t, \alpha) \\
+ \left( \ln \frac{\hat{\beta}}{\beta} - 1 \right) \sum_{l=1}^{n} N^l(t) = -\ln \gamma. \hspace{1cm} (15)
\]

Where

\[
D^l(t, \alpha) = \sum_{k=1}^{N_1(t)} \left\{ \ln \left( 1 + \frac{x_{l,2k-1}}{\alpha} \right) + \sum_{k=1}^{N_2(t)} \left\{ \ln \left( 1 + \frac{x_{l,2k}}{\alpha + cx_{l,2k-1}} \right) \right\} + 1\{K(t) = 0\} \left\{ \ln \left( 1 + \frac{t - t_{1,N_1(t)}}{\alpha} \right) \right\} + 1\{K(t) = 1\} \left\{ \ln \left( 1 + \frac{t - t_{1,N_1(t)}}{\alpha + cx_{1,N_1(t)}} \right) \right\}.
\]

7 Simulation study

In this section we show how to apply the previous theoretical results to the lifetime data. It is devoted to introduce numerical results based on a large simulation study. The simulation has been made by writing some computer programs. In such a study, estimation of the parameters of the Pareto intensity of the second kind and of the Fisher information are developed. The confidence regions for the parameters based on likelihood ratio statistic are represented.

**Example 7.1** Parameter estimator of \( \alpha \) and \( \beta \) with known \( c = 0.5 \) are illustrated in this example. A sample with size \( n = 100 \) was observed until time \( t = 10 \). Let \( \alpha = 0.5 \), \( \beta = 2 \) and \( s = 100 \), where \( s \) is the number of simulations.

Simulations are carried out with different sets of parameters. From one set of parameters, the bias and variance of the estimators are estimated by their empirical version on 100 replicates. The estimations are given in table 7.1.

<table>
<thead>
<tr>
<th></th>
<th>( \hat{\alpha} )</th>
<th>( \hat{\beta} )</th>
<th>( I_{1,1}(\hat{\theta}) )</th>
<th>( I_{1,2}(\hat{\theta}) )</th>
<th>( I_{2,2}(\hat{\theta}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Estimation</strong></td>
<td>0.5062</td>
<td>2.0164</td>
<td>4091.3525</td>
<td>-1249.4025</td>
<td>538.2513</td>
</tr>
<tr>
<td><strong>Empirical mean</strong></td>
<td>0.5487</td>
<td>2.0125</td>
<td>4091.4910</td>
<td>-1244.2095</td>
<td>535.6750</td>
</tr>
<tr>
<td><strong>Empirical variance</strong></td>
<td>16.8125 (10^{-4})</td>
<td>53.5701 (10^{-4})</td>
<td>9655.4974</td>
<td>1516.9193</td>
<td>169.6561</td>
</tr>
</tbody>
</table>

Estimation results considering an average of 100 simulations
The parameter estimators of $\alpha$ and $\beta$ are respectively $\hat{\alpha} = 0.5062$ and $\hat{\beta} = 2.0164$. We obtain then the following estimation of the Fisher information matrix.

$$
\hat{I}(\hat{\theta}) = 
\begin{pmatrix}
4091.3525 & -1249.4025 \\
-1249.4025 & 538.2513
\end{pmatrix}
$$

If we inverse this matrix, we get in the first diagonal the variance of the parameter estimators:

\begin{align*}
\text{Var}(\hat{\alpha}) &= 8.3948 \times 10^{-4} \\
\text{Var}(\hat{\beta}) &= 6.38108 \times 10^{-4}
\end{align*}

Therefore, if we use the equation (13), the approximation 95\% two side confidence intervals of the parameters $\alpha$ and $\beta$ are $[0.4494, 0.5629]$ and $[1.8598, 2.1729]$ respectively.

In the following simultaneous confidence region based on likelihood ratio statistic for $\hat{\alpha}$ and $\hat{\beta}$ is represented in figure 7.1, where $\gamma = 0.05$. If we use a sample with size $n = 200$ and if $\gamma = 0.05$ we obtain then in figure 7.1 the confidence region based on likelihood ratio statistic for $\hat{\alpha}$ and $\hat{\beta}$.

Simultaneous confidence region based on the likelihood ratio for $\hat{\alpha} = 0.5062$, $\hat{\beta} = 2.0164$ and $n = 100$
Simultaneous confidence region based on the likelihood ratio for $\hat{\alpha} = 0.5562$, $\hat{\beta} = 1.9518$ and $n = 200$

Based on the previous analysis, we can remark that the empirical variance of $\hat{\alpha}$ is greater than the variance of $\hat{\alpha}$ and the empirical variance of $\hat{\beta}$ is smaller than the variance of $\hat{\beta}$.

The mean squared errors (MSE) of $\hat{\alpha}$ and $\hat{\beta}$ are given in table 1:

<table>
<thead>
<tr>
<th>$s$</th>
<th>250</th>
<th>500</th>
<th>750</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>MSE(\hat{\alpha})</td>
<td>0.005901</td>
<td>0.005314</td>
<td>0.005152</td>
<td>0.004835</td>
</tr>
<tr>
<td>MSE(\hat{\beta})</td>
<td>0.005168</td>
<td>0.004902</td>
<td>0.004826</td>
<td>0.004753</td>
</tr>
</tbody>
</table>

Table 1: MSE of $\hat{\alpha}$ and $\hat{\beta}$

Based on the results booked on the table 1, we could conclude that:

1. If the number of simulations $s$ increases then the mean squared errors of $\hat{\alpha}$ and $\hat{\beta}$ decreases.
2. For \( s = 1000 \) the mean squared errors of \( \hat{\alpha} \) and \( \hat{\beta} \) will be very small.

Based on the previous analysis and on the results of figure 7.1 and figure 7.1, we could say that:

1. The simultaneous confidence region based on the likelihood ratio for \( n = 200 \) is smaller than for \( n = 100 \).

2. The simultaneous confidence region based on the likelihood ratio will be smaller with increasing sample size \( n \).

8 Conclusion

Confidence regions for parameters of lifetime distributions are of interest in the view of theory and application. If the parameters of the lifetime distribution are estimated it is possible to calculate for instance the reliability of the product. Therefore, it is useful to find simultaneous confidence estimations. In this paper we discussed the parameter estimation of an alternating repair model using a Pareto intensity of second kind. The maximum likelihood estimators, the estimation of the Fisher information matrix and the confidence regions based on the likelihood ratio statistics are obtained. The theoretical results presented in this paper have been applied on sets of simulated data.

References


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