A New Approach for Solving
an Optimization Problem

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Abstract

In this paper, we propose an algorithm of an interior point methods to solve a linear complementarity problem (LCP). The study is based on the transformation of a linear complementarity problem (LCP) into a convex quadratic problem, then we use the linearization approach for obtain the simplified problem of Karmarkar. Theoretical results deduct of those established later.

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1 Introduction

The linear complementarity problem (LCP) generalizes the Mathematical Programming (Convex quadratic programming, Linear programming, Positive semidefinite programming... etc.), it is written as:

\[
\begin{cases}
\text{find } (x, y) \in \mathbb{R}^{n+m} \text{ such that } \\
x^t y = 0, \
y = Mx + q, \\
(x, y) \geq 0.
\end{cases}
\]  

(LCP)

where \( M \in \mathbb{R}^{(n+m)\times(n+m)} \) is a matrix, \( q \in \mathbb{R}^{n+m} \).

The principal idea of this method to replace a linear complementarity problem by a convex quadratic program. After the appearance of the Karmarkar’s
algorithm [4], the researchers introduced extensions for the convex quadratic programming [3][1][9][6][7]. We propose in this work an interior point method of type projectif to resolve a more general problem where the objective is not inevitably linear. We combine the approach of linearization with ingredients brought by Karmarkar.

The paper is organized as follows. In the next section, the statement of the problem is presented. In section 3, we deal with the new method for solving the linear complementarity problem and description of the algorithm. In the section 4, we study the convergence of our algorithm. In section 5, a conclusion and remarks are given.

We use the classical notation. In particular, $\mathbb{R}^{n+m}$ denotes the $(n+m)$-dimensional Euclidean space. Given $u, v \in \mathbb{R}^{n+m}$, $u^t v = \sum_{i=1}^{n+m} u_i v_i$ is their inner product, and $\|u\| = \sqrt{u^t u}$ is the Euclidean norm. Given a vector $z \in \mathbb{R}^{n+m}$, $D = \text{diag}(z)$ is the $(n+m) \times (n+m)$ diagonal matrix. $I$ is the identity matrix and $e$ is the identity vector.

2 Presentation of the problem

The linear complementarity problem associated with the convex nonlinear programming is written as follows:

\[
\begin{aligned}
\text{min} & \quad \sum_{i=1}^{n+m} u_i v_i \\
\text{s.t.} & \quad Mz + q \\& \quad w \geq 0, \quad z \geq 0.
\end{aligned}
\]

(LCP)

where $w \in \mathbb{R}^{n+m}$, $M \in \mathbb{R}^{(n+m) \times (n+m)}$ is a matrix, $q \in \mathbb{R}^{n+m}$.

Remark 1 In general, we can not transform an arbitrary linear complementarity problem in a convex quadratic program unless the matrix $M$ is positive semidefinite.

Theorem 1 A linear complementarity problem is equivalent to the following convex quadratic program:

\[
\begin{aligned}
\text{min} & \quad z^t (Mz + q) = \min \quad z^t Mz + z^t q = 0 \\
\text{s.t.} & \quad Mz + q \geq 0, \quad z \geq 0.
\end{aligned}
\]

(1)

where $(z^*, Mz^* + q)$ is a solution of the linear complementarity problem if and only if $z^*$ is a optimal solution of convex quadratic program (1) with $(z^*)^t (Mz^* + q) = 0$. 
In the next section we have introduced Karmarkar’s algorithm for solving the linear complementarity problem \((LCP)\).

3 A new algorithm for solving \((LCP)\).

We can write the problem (1) under the following simplified Karmarkar’s form:

\[
\begin{align*}
\min & \quad g(t) \\
\text{s.t} & \quad Bt = 0, \ t \in S_{n+m+1}.
\end{align*}
\]

where \(g : \mathbb{R}^{n+m+1} \to \mathbb{R}\) is a nonlinear, convex and differentiable function.

\(S_{n+m+1} = \{ t \in \mathbb{R}^{n+m+1} : e^t_{n+m+1}t = 1, t \geq 0 \}\) is the simplex of dimension \(n + m\) and of the center \(a_i = \frac{1}{n+m+1}, \ i = 1, \ldots, n + m + 1\).

We introduce the projective Karmarkar’s transformation defined by:

\[
T_k : \mathbb{R}^{n+m} \to S_{n+m+1}
\]

\[
z \to t,
\]

where

\[
\begin{align*}
t_i &= \frac{\hat{z}_i}{\hat{z}_{n+m}}, \ i = 1, \ldots, n + m \\
&\quad \frac{1 + \sum_{i=1}^{n+m} t_i}{n+m+1}
\end{align*}
\]

\[
t_{n+m+1} = 1 - \sum_{i=1}^{n+m} t_i.
\]

and we have \(z = T_k^{-1}(t) = \frac{D_k t[n+m]}{t_{n+m+1}}, \) where \(t[n+m] = (D_k^{-1}z)t_{n+m+1} = (z_i)_{i=1}^{n+m}, \ D_k = diag(z_k)\).

Thus the problem

\[
\begin{align*}
\min & \quad f(z) = \min \ z'(Mz + q) \\
\text{s.t} & \quad Mz = l, \ l \geq 0.
\end{align*}
\]

is transformed as follows
\[
\begin{align*}
\min_{s.t.} \quad & f(T_k^{-1}(t)) = \min f(D_k t[n + m]) \\
M_{k} D_k t[n + m] = l, \\
\sum_{i=1}^{n+m+1} t_i = 1, \\
t[n + m] \geq 0, \quad t_{n+m+1} \geq 0.
\end{align*}
\]

Hence, it is advisable to write (3) under the equivalent form:

\[
\begin{align*}
\min_{s.t.} \quad & g(t) = \min t_{n+m+1} f(D_k t[n + m]) \\
M_k t = 0, \quad t \in S_{n+m+1}.
\end{align*}
\]

where \(M_k = [MD_k, -l], \quad t = \begin{bmatrix} t[n + m] \\ t_{n+m+1} \end{bmatrix} \)

Note that the optimal value of \(g\) is zero and the center of the simplex is feasible for (4), also note that the function \(g\) is convex on the set: \(\{t \in \mathbb{R}^{n+m+1} : M_k t = 0\} \).

Applying the linearization of the function \(g\) in the neighborhood of the center of the simplex \(a\), and by introduce a ball of center \(a\) considered as a neighborhood of \(a\), we have \(g(t) = g(a) + \langle \nabla g(a), t - a \rangle\), for all \(t \in \{t \in \mathbb{R}^{n+m+1} : \|t - a\|^2 \leq \beta^2\} \).

Then we have the following sub-problem:

\[
\begin{align*}
\min_{s.t.} \quad & \nabla g(a)^t t \\
M_k t = 0, \\
e_{n+m+1} t = 1, \\
\|t - a\|^2 \leq \beta^2
\end{align*}
\]

**Lemma 2** The optimal solution of the problem (5) is explicitly given by \(t^k = a - \beta d^k\), where \(d^k = \frac{p_k}{\|p_k\|}, \quad P_k = p_B \nabla g(a), \quad B_k = \begin{bmatrix} M_k \\ e_{n+m+1} \end{bmatrix} \).

**Proof.** We put \(z = t - a\), then we have \(B_k z = \begin{bmatrix} M_k \\ e_{n+m+1}^t \end{bmatrix} (t - a) = 0\), and the sub-problem (5) is equivalent

\[
\begin{align*}
\min_{s.t.} \quad & \nabla g(a)^t z \\
B_k z = 0, \\
\|z\|^2 \leq \beta^2.
\end{align*}
\]
$z^*$ is a solution of (6) if and only if $\exists \lambda \in \mathbb{R}^{n+m+1}$, $\exists \mu \geq 0$ such that:

$$\nabla g(a) + B_k^t \lambda + \mu z^* = 0 \quad \text{.....(7)}$$

Multiplying both members of (7) by $B_k$ we obtained:

$$B_k \nabla g(a) + B_k B_k^t \lambda + \mu B_k z^* = 0 \iff B_k \nabla g(a) + B_k B_k^t \lambda = 0$$

then

$$\lambda = -(B_k B_k^t)^{-1}(B_k \nabla g(a))$$

by substituting in (7)

$$z^* = -\frac{1}{\mu} P_k,$$

where $P_k = [I - B_k B_k^t (B_k B_k^t)^{-1} B_k] \nabla g(a)$,

$$\|z^*\| = \frac{1}{\mu} \|P_k\| = \beta \Rightarrow z^* = -\beta \frac{P_k}{\|P_k\|} = -\beta d^k,$$

and we have

$$t^k = t^* = a + z^* = a - \beta d^k.$$  

from where the result

\[ \square \]

### 3.1 Description of the algorithm

Initialization: $\varepsilon > 0$, $z^0$: is a strictly feasible point.

Begin

while $(f(z^k) \geq \varepsilon)$ do

Compute the matrices:

$$D_k = \text{diag}(z^k), \ M_k = [MD_k, -I], \ B_k = \left[ \begin{array}{c} M_k \\ e_{n+m+1} \end{array} \right]$$

Comput:

$$P_k, \ d^k, \ t^k = a - \beta d^k.$$  

Comput:

$$z^{k+1} = T_k^{-1}(t^k).$$

$k = k + 1.$

end while

END.

### 4 Convergence of algorithm

To established the convergence of our algorithm, we introduce a potential function associated with problem (1), and defined by:

$$F(z) = (n + m + 1) \log(f(z) - f(z^*)) - \sum_{i=1}^{n+m} \log(z_i)$$

we have the following lemma
Lemma 3  For each iteration we obtain a reduction of the function $g$ i.e: $g(t^k) \leq g(a)$.

Proof. we have:

$$g(z^k) = g(a) + \langle \nabla g(a), z^k - a \rangle,$$
and we have: $t^k = a - \beta \frac{P_k}{\|P_k\|}$, then

$$g(z^k) - g(a) = \langle \nabla g(a), -\beta \frac{P_k}{\|P_k\|} \rangle$$
$$= -\beta \frac{P_k}{\|P_k\|} \langle \nabla g(a), P_k \rangle$$
$$= -\beta \frac{P_k}{\|P_k\|} \|P_k\|^2 < 0$$

from where the result

Theorem 4  In every iteration of our algorithm, potential function is reduced of a constant value such that:

$$F(z^{k+1}) < F(z^k) - \delta$$

Proof.

We have:

$$F(z^{k+1}) - F(z^k) = (n + m + 1) \log \left[ \frac{f(z^{k+1}) - f(z^*)}{f(z^k) - f(z^*)} \right] - \sum_{i=1}^{n+m} \log \left( \frac{z^k_i}{z^*_i} \right)$$
$$= (n + m + 1) \log \frac{g(t^k)}{g(a)} - \sum_{i=1}^{n+m} \log(t^k_i)$$
$$\leq (n + m + 1) \log(1 - \frac{\beta}{(n+m+1)} + \frac{\beta^2}{2(1-\beta)^2})$$
$$\leq -\beta + \frac{\beta^2}{2(1-\beta)^2}$$

from where the result

Under hypotheses follows:

1. We have verifying realisable solution: $z^0 \geq 2^{-2L}e_{n+m+1}$
2. The optimal solution $z^*$ verifies $z^* \leq 2^{2L}e_{n+m+1}$, for any solution $z$ we have $-2^{3L} \leq f(z^*) \leq 2^{3L}$.

we have the following theorem

Theorem 5  For each iteration, the algorithm finds the optimal solution after $O((n + m + 1)L)$ iterations.
Proof. We have

\[
\frac{f(z_k) - f(z^*)}{f(z^0) - f(z^*)} = \eta(z^k) \exp \left[ \frac{F(z_k) - F(z^0)}{n + m + 1} \right]
\]

under hypotheses (1) and (2) we have: \( \eta(z^k) \leq 2^{2L} \) then:

\[
f(z_k) - f(z^*) \leq 2^{2L}(f(z^0) - f(z^*)) \exp \left[ \frac{F(z_k) - F(z^0)}{n + m + 1} \right]
\]

\[
\leq 2^{2L} 2^{3L} \exp \left( \frac{-k\delta}{n + m + 1} \right)
\]

hence,

\[k \geq \xi(n + m + 1) L, \text{ where } \xi \in \mathbb{R}^*_+.
\]

from where the result \( \blacksquare \)

5 Concluding remarks

In this paper, we presented an interior point method for resolution of a linear complementarity problem \((LCP)\). We tried to take advantage of ideas of the projective method of Karmarkar applied to a linear problem under special form. To have to apply these technique, we used the transformation of the \((LCP)\) to the convex quadratic problem, idea of the method of linearization for the obtaining of a linear problem which approximates the original problem of Karmarkar. Finally, the numerical test is an interesting topic for investigating the behavior of the algorithm so as to be compared with other approaches.

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