Finding a Strict Feasible Dual Initial Solution for Quadratic Semidefinite Programming

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Abstract
In this paper, we propose an algorithm to compute the dual initial feasible solution for solving a primal-dual optimization problem (convex quadratic semidefinite programming) by an interior point method.

Mathematics Subject Classification: 90C25, 90C30, 90C51, 90C20

Keywords: Quadratic programming, Convex nonlinear programming, Interior point methods.

1 Introduction

The convex quadratic semidefinite programming (CQSDO) is a model which traduces many real applications. In term of research, it is one of subject treated with fervour, in particular the problem of initialization in optimization problem [1][2][3][5][7][6][4]. Choice of starting primal and dual point is an important practical issue with a significant effect on the robustness of the algorithm. A poor choice \((X^0, y^0, S^0)\) satisfying only the minimal conditions \(X^0 \succ 0\), \(S^0 \succ 0\). In interior point methods, the successive iterates should be strictly feasible. In consequence, a major concern is to find an initial primal feasible solution \(X^0\). The object of this paper is to find the initial dual feasible solution \((y^0, S^0)\).

Some of the notations used throughout the paper are as follows: \(\mathbb{R}^n\), \(\mathbb{R}_+^n\) and \(\mathbb{R}_{++}^n\) denote the set of vectors with \(n\) components, the set of nonnegative
vectors and the set of positive vectors, respectively. $\mathbb{R}^{n \times n}$ denotes the set of $n \times n$ real matrices. $\| \cdot \|_F$ and $\| \cdot \|_2$ denote the Frobenius norm and the spectral norm for matrices, respectively. $S_n$, $S_n^+$ and $S_n^{++}$ denote the cone of symmetric, symmetric positive semi-definite and symmetric positive definite $n \times n$ matrices, respectively. We use the matrix inner product $A \cdot B = \text{Tr}(A^t B)$.

In this paper, we consider the convex quadratic semidefinite optimization problem ($CQSDO$) in its standard form:

$$\begin{align*}
\min & \quad \frac{1}{2} X \cdot \Omega(X) + C \cdot X \\
\text{s.t} & \quad A_i \cdot X = b_i, \ i = 1, \ldots, m. \\
& \quad X \succeq 0.
\end{align*}$$

$(P)$

and its dual problem is

$$\begin{align*}
\max & \quad b' y - \frac{1}{2} X \cdot \Omega(X) \\
\text{s.t} & \quad \sum_{i=1}^{m} y_i A_i + S = C + \Omega(X), \\
& \quad S \succeq 0, \ y \in \mathbb{R}^m.
\end{align*}$$

$(D)$

where $\Omega(X) : S^n \to S^n$ is a given self-adjoint positive semidefinite linear operation on $S^n$, i.e., for any $A, B \in S^n$, then $\Omega(A) \cdot B = A \cdot \Omega(B)$ and $\Omega(A) \cdot A \succeq 0$. To simplify matters we will restrict ourselves to the following special case $\Omega(X) = \sum_{i=1}^{l} H_i^t X H_i$ where $H_i$ is a matrix in $\mathbb{R}^{n \times n}$ and $l$ is an integer not greater than $n^2$.

The strict feasibility problem of $(P)$ and $(D)$ is to find $(X, y, S)$, where the strict feasible primal solution $X$ is computed using the following problem:

$$\{ X \in S_n^+, \ A_i \cdot X = b_i, \ i = 1, \ldots, m \}$$

To calculate the strict feasible dual solution, using the same help of the primal solution[4], but the main drawback is the vector $y$, because the algorithm we used is applicable only for positive elements.

We set $y_i = y_i^+ - y_i^-$. Then, the problem of dual feasibility becomes as follows:

$$\left\{ \left( y^+, y^-, S \right) \in \mathbb{R}_+^m \times \mathbb{R}_+^m \times S_n^+, \ \sum_{i=1}^{m} y_i^+ A_i - \sum_{i=1}^{m} y_i^- A_i + S = C + \Omega(X), \ (y^+_i, y^-_i) > 0, \ S \in S_n^+ \right\}$$

$(df)$
On way to solve a strictly feasible problem consists in introducing an additional variable $\lambda$ as follows:

\[
\begin{aligned}
\min & \quad \lambda \\
\text{s.t} & \quad \sum_{i=1}^{m} y_i^+ A_i - \sum_{i=1}^{m} y_i^- A_i + S + \lambda (C + \Omega(X)) - \sum_{i=1}^{m} y_i^{0+} A_i + \sum_{i=1}^{m} y_i^{0-} A_i - S^0 = C + \Omega(X), \\
y_i^+ > 0, y_i^- > 0, S > 0, \lambda \geq 0.
\end{aligned}
\]

where $(y_i^{0+}, y_i^{0-}, S^0) \in \mathbb{R}_+^m \times \mathbb{R}_+^m \times S_n^+$ is arbitrary.

The problem $(Pa)$ can be written as a following linear semidefinite program:

\[
\begin{aligned}
\min & \quad C' \cdot T' \\
\text{s.t} & \quad A_i' \cdot T' = C + \Omega(X), \\
& \quad T' \succ 0.
\end{aligned}
\]

where $C'$ is the $(2m + n + 1) \times (2m + n + 1)$ symmetric matrix defined by:

\[
C'[i, j] = \begin{cases} 
1 & \text{if } i = j = 2m + n + 1 \\
0 & \text{otherwise}
\end{cases}
\]

and $A_i'$ is the $(2m + n + 1) \times (2m + n + 1)$ symmetric matrix defined by:

\[
A_i' = \begin{bmatrix}
A_i & 0 & 0 & 0 \\
0 & -A_i & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & C + \Omega(X) - \sum_{i=1}^{m} y_i^{0+} A_i + \sum_{i=1}^{m} y_i^{0-} A_i - S^0
\end{bmatrix}
\]

and $T'$ is the $(2m + n + 1) \times (2m + n + 1)$ symmetric matrix defined by:

\[
T' = \begin{bmatrix}
Y^+ & 0 & 0 \\
0 & Y^- & 0 \\
0 & 0 & S \\
0 & 0 & 0 & \lambda
\end{bmatrix}
\]

where

\[
Y^+ \text{ is the } (m \times m) \text{ matrix defined by: } Y^+ = \begin{bmatrix}
y_1^+ & 0 & \ldots & 0 \\
\times & \ldots & \ldots \\
\times & \ldots & \ldots \\
y_m^+ & 0 & \ldots & 0
\end{bmatrix}
\]
\[ Y^- \text{ is the } (m \times m) \text{ matrix defined by: } Y^- = \begin{bmatrix} \ y_1^- & 0 & \ldots & 0 \\ \times & \ldots & \times \\ \times & \ldots & \times \\ \ y_m^- & 0 & \ldots & 0 \end{bmatrix} \]

**Lemma 1 [5]:** \( T^* \) is a solution of problem \((df)\) if and only if \( \begin{bmatrix} T^* & 0 \\ 0 & \lambda \end{bmatrix} \) is an optimal solution of \((LSDP)\), with \( T^* \succ 0 \), and \( \lambda \) sufficiently small (\( \lambda \leq \varepsilon \)).

To compute the optimal solution of \((LSDP)\), we use only the second phase of the projective interior point method. The corresponding algorithm is follows:

### 2 Algorithm for solving \((LSDP)\)

**Description of the algorithm**

- **(a) Initialization:** \( \varepsilon = 10^{-8} \), \( \lambda_0 = 1 \), \( T_0 = \begin{bmatrix} 1 & 0 & 0 \\ \ldots & \ldots & \ldots \\ 1 & 0 & 0 \\ 0 & \ldots & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \),

- **(b)** \( k = 0 \)

**If** \( \max \left\| C + \Omega(X) - \sum_{i=1}^{m} y_k^+ A_i + \sum_{i=1}^{m} y_k^- A_i - S_k \right\| \leq \varepsilon \). **Stop,** \( T_k \) is an \( \varepsilon \)-approximate solution of \((df)\)

**else**

- **(b) If** \( \lambda_k \leq \varepsilon \), stop \( T_k \) is an \( \varepsilon \)-approximate solution of \((df)\)

**else**

- **(c) Step** \( k \)

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1. set \( z = C' \cdot T'_k \)
2. determine $L_k$, such that: $T'_k = L_k L_k^t$ (Cholesky decompozition) and we compute
   \[ C_k = L_k C' L_k \]
   \[ A_i^k = L_k A_i^t L_k, \quad i = 1, \ldots, 2m + n + 1. \]

3. Compute the matrix $M$ and the vector $d$ by:
   \[ M_{ij} = A_i^k \cdot A_j^k + C + \Omega(X), \quad i, j = 2m + n + 1 \]
   \[ d_i = -(C + \Omega(X)) z_k - C_k \cdot A_i^k, \quad i = 1, 2m + n + 1 \]

4. Solve the linear system
   \[ M u = d \]

5. Compute
   \[ V_k = C_k + \sum_{i=1}^{2m+n+1} u_i A_i^k \]
   \[ u_k = -\sum_{i=1}^{2m+n+1} (C + \Omega(X)) u_i - z_k \]
   \[ \tau = (\|V_k\|^2 + v_k^2)^{1/2} \]
   \[ \lambda = \frac{1}{(2m+n+1)\tau} \text{tr}(V_k) \]
   \[ \sigma = \frac{1}{(2m+n+1)\tau} \|V_k\|^2 - \lambda^2 \]
   \[ \beta_k = \zeta(\max(\frac{\tau}{\lambda}, \lambda + \sigma \sqrt{2n + m}))^{-1} \]

6. Compute
   \[ T'_{k+1} = T'_k - \frac{\beta_k}{\tau \beta_k v_k} L_k (V_k - v_k I) L_k^t, \quad \text{with} \quad T'_{k+1} = \begin{bmatrix} T_{k+1} & 0 \\ 0 & \lambda_{k+1} \end{bmatrix} \]

7. Take $k = k + 1$ and go back to (b).

3 Conclusion

In this paper, we presented an algorithm for compute the strictly feasible dual initial solution of a convex quadratic semidefinite problem. We tried to take advantage of ideas of the projective method of Karmarkar applied to a linear semidefinite problem under special form. The implementation of the algorithm remains which will require certain arrangements taking into account the best developments relative to the numerical performances.
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Received: November, 2011