Abstract

In this work we present the mathematical analysis of a system modeling ion migration through biological membranes. The model includes both the effects of biochemical reaction between ions and fixed charges. The model is a nonlinear coupled system. In the first we describe the mathematical model. To develop the mathematical analysis of our model, we define an approximating scheme and by using Schauder fixed point theorem in ordered Banach spaces, we show the existence of a solution for this approached problem. Finally by making some estimations we prove that the solution of the truncated system converge to the solution of our problem.

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1 Introduction

In this paper we consider a class of models of ions migration through biological membranes. Such migrations exist for most living cells and some biochemical processes. The motion of ions is supposed due to diffusion and to the effect of the electrical field. Furthermore ions can undergo reactions. So the ions concentrations satisfy the Nernst-planck equations,
including a kinetic reaction terms and the potential is given by poisson equation. These equations are:

\[
\begin{align*}
\frac{\partial C_i}{\partial t} - d_i \Delta C_i - m_i \text{div}(C_i \nabla \phi) &= F_i(C, \phi) \quad \text{for } i=1, \ldots, N_s \\
-\varepsilon \Delta \phi &= \frac{\sum_{i=1}^{N_s} z_i C_i}{1 + \varepsilon \sum_{i=1}^{N_s} C_i} - f
\end{align*}
\]

(1)

For each \( i \), \( C_i \) is the concentration of the \( i \) species which has mobility \( d_i \) and valency \( z_i \). \( \phi \) is the electrical potential, \( f \) is the fixed charges concentration and \( F_i \) is the reaction terms. We suppose that \( F_i \) depends continuously on the \( C_j \)'s and \( \phi \), and that \( f \) is a bounded function. We suppose that \( d_i \) is a positive constant for each \( i \).

These models for passive migration (i.e. without reaction) have been deeply studied in the biophysical literature, in order to explain the behaviour of ionic currents through biological membranes, (see Lakhshminarayanaiah [7], Mackey [5]). Two simplifications of these equations have been quite popular in this literature, namely the Goldman hypothesis where the electrical field is supposed to be constant inside the membrane and the electroneutral hypothesis where the neutrality at each point of the membrane is assumed (see for example Mackey [5]). It has been recognized by McGillivray [1] that these models are the limit of the full equations when the ratio \( \sqrt{\varepsilon} = \frac{\lambda}{l} \) of the Debye length to the membrane thickness goes to, respectively, infinity or zero. Here we study the second case. Often enzymes are fixed to biological membranes and ions undergo biochemical reactions when crossing the membrane. Valleton [3] did a general biophysical of coupling of electromigration diffusion with biochemical reactions.

In this paper we present a mathematical study of such systems, for a large class of reaction kinetics, including the usual biochemical kinetics as the Michaelis-Menton one (a previous work on the one dimensional and stationary case was done by Henry and Louro [2]). This article is organized in the following way. In the next section we describe the mathematical model. The third section is devoted to mathematical analysis of our model. We define an approximating scheme of our model and by using Schauder fixed point theorem in ordered Banach spaces, we show the existence of a solution for this approached problem. Finally, by making some estimations we prove that the solution of the truncated system converge to the solution of our problem.
2 Modeling the problem

Let us consider the membrane which fills the bounded open set $\Omega$ of $IR^N$. This type of reaction within the membrane always contain electroactive ions $(A_i)_{1 \leq i \leq N_s}$ as one of their major components. The movement of ions is supposed to be due to diffusion and also the effect of electrical field. The mass conservation equation for the species $A_i$ is

$$\frac{\partial C_i}{\partial t} + \text{div}(J_i) = F_i$$

where $C_i$ is the concentration of species $A_i$, $F_i$ denotes the production rate of $A_i$ due to all the homogeneous reactions in which it is involved and $J_i$ is its molar transport flux. Consequently, migration is included along with diffusion as possible modes of transport for each species. The molar flux $J_i$ then becomes

$$J_i = -d_i \nabla C_i - m_i C_i \nabla \phi$$

$$m_i = \frac{d_i z_i F_a}{RT_e}$$

where $z_i F_a$ is the charge carried by a mole of species $A_i$, $R$ is the universal gas constant and $T_e$ is the local temperature. The transport equation for each species becomes

$$\frac{\partial C_i}{\partial t} - d_i \Delta C_i - m_i \text{div}(C_i \nabla \phi) = F_i(C_1, ..., C_{N_s})$$

The species must also satisfy the electro-neutrality condition everywhere in the system, i.e.

$$-\varepsilon \Delta \phi = \frac{\sum_{i=1}^{N_s} z_i C_i}{1 + \varepsilon \sum_{i=1}^{N_s} C_i} - f$$

$$\begin{cases} 
\frac{\partial C_i}{\partial t} - d_i \Delta C_i - m_i \text{div}(C_i \nabla \phi) = F_i(C) \text{ in } Q_T, \text{ for } i = 1, ..., N_s \\
-\varepsilon \Delta \phi = \frac{\sum_{i=1}^{N_s} z_i C_i}{1 + \varepsilon \sum_{i=1}^{N_s} C_i} - f \quad \text{ in } Q_T \\
d_i \frac{\partial C_i}{\partial \nu} + m_i C_i \frac{\partial \phi}{\partial \nu} = 0 \quad \text{ in } \Sigma_T \\
\phi(t, x) = 0 \quad \text{ in } \Sigma_T \\
C_i(0, x) = C_{i,0}(x) \quad \text{ on } \Omega \\
\phi(0, x) = \phi_0(x) \quad \text{ on } \Omega 
\end{cases}$$

(5)
where $Q_T = ]0, T[ \times \Omega$, $\sum_T = ]0, T[ \times \partial \Omega$ and $T > 0$.

We consider a set of biochemical reactions with one substrate $S$ and some product $P$, and suppose that the reaction kinetics are of the form

$$F_i(C_1, \ldots, C_{Ns}) = 0 \text{ for } i \in A_0$$
$$F_i(C_1, \ldots, C_{Ns}) = G(C_1, \ldots, C_{Ns}) \text{ for } i \in A_p$$
$$F_i(C_1, \ldots, C_{Ns}) = -G(C_1, \ldots, C_{Ns}) \text{ for } i \in A_s$$

where $A_0 \cup A_p \cup A_s = [1, \ldots, Ns]$ and $G : IR^{Ns} \to [0, +\infty[$ is a continuous function.

3 Mathematical analysis of the problem

3.1 Position problem

We suppose the following hypothesis:

$G : IR^{Ns} \to [0, +\infty[$, measurable and quasi-positive namely, let $U$ be a vector then let’s note by $\hat{U}_i$ the vector obtained by replacing the component number $i$ by 0, then we suppose the following assumptions. For every $U \geq 0$

$$\begin{cases}
G(U) \geq 0 \\
G(\hat{U}_i) = 0 \text{ for } i \in A_s
\end{cases} \quad (6)$$

$G : IR^{Ns} \to IR$ continuous and locally lipschitz, namely

$$|G(r) - G(\hat{r})| \leq K(R)(|r - \hat{r}|)$$
for all $0 \leq |r|, |\hat{r}| \leq R \quad (7)$

for all $i = 1, \ldots, Ns \quad C_{i,0} \in L^2(\Omega)$ and satisfy $C_{i,0} \geq 0 \quad (8)$

Before stating the main result of this part, we have to clarify in which sense we want to solve problem (5).

**Definition 3.1** $(C, \phi) = (C_1, C_2, \ldots, C_{Ns}, \phi)$ is a weak solution of (5) if and only if for every $1 \leq i \leq Ns$
\[ C \in C([0, T], L^2(\Omega)^{N_s}) \cap L^2(0, T, H^1(\Omega)^{N_s}), \phi \in L^\infty (0, T, H^1_0(\Omega)), F_i(C) \in L^1(Q_T) \]

\[ \text{for every } v \in C^1(Q_T) \text{ such that } v(T, .) = 0 \]

\[ \int_{Q_T} (-C_i \frac{\partial v}{\partial t} + d_i C_i \nabla v + m_i C_i \nabla \phi \nabla v) - \int_\Omega C_{i,0}(x) v(0, x) = \int_{Q_T} F_i(C) v \]

\[ \text{for all } \Psi \in H^1_0(\Omega) \]

\[ \epsilon \int_\Omega \nabla \phi \nabla \Psi = \int_\Omega (\sum_{i=1}^{N_s} z_i C_i + f) \Psi \quad a.e \ t > 0 \]

\[ \phi(0, x) = \phi_0(x) \quad \text{on } \Omega \]

(9)

3.2 Main result:

The main result of this paper is the following theorem.

**Theorem 3.2** We suppose that the hypothesis (6), (7) and (8) are satisfied, then the problem (5) admits a weak solution \((C, \phi)\) satisfying \(C \geq 0\) in \(Q_T\) and \(\phi \in L^\infty (0, T; W^{1,\infty}(\Omega))\).

3.3 proof of the main result:

3.3.1 approximating scheme

We associate to every function \(F_i\) the function \(\widehat{F}_i\) such that \(\widehat{F}_i(C) = F_i(C^+)\) and we consider the function of truncation \(\eta_n \in C^\infty_0(\mathbb{IR}^{N_s})\) that satisfies

\[ \eta_n \geq 0, \eta_n \leq 1 \]

\[ \eta_n(r) = 1 \quad \text{if } |r| \leq n \]

\[ \eta_n(r) = 0 \quad \text{if } |r| \geq n + 1 \]

we define for every \(C = (C_1, C_2, ..., C_{N_s}) \in \mathbb{IR}^{N_s}\).

\[ F_i^n(C) = \eta_n(|C|) \widehat{F}_i(C) \]

(10)

Note in passing that \(F_i^n\) as defined, is globally lipschitz. Indeed, \(F_i^n\) is locally lipschitz from (7), and as it’s bounded by definition, then \(F_i^n\) is globally lipschitz. Furthermore, let’s consider for every \(\psi \in C([0, T]; H^1(\Omega))\) and \(t \in [0, T]\), the bilinear form \(a^i_\psi(t, ...,)\) defined on \(H^1(\Omega) \times H^1(\Omega)\) by

\[ a^i_\psi(t, u, v) = \int_\Omega (d_i \nabla C_i \nabla v + m_i u \nabla \psi \nabla v + \lambda uv) \]
Where $\lambda$ is a strictly positive real whose value will be set later. Finally, we introduce for $v \in L^2(Q_T)$ its regularization time

$$v^{(n)}(t, x) = \int_\Omega n v(s, x) \exp(n(s - t)) ds$$

From [4], we deduce that

$$\begin{cases}
    v^{(n)}(t, x) \in C([0, T]; L^2(\Omega)) \\
v^{(n)} \to v \text{ in } L^2(Q_T) \\
\left|\sup_{0 < t < T} \|v^{(n)}(t, \cdot)\|_{L^1(\Omega)}\right| \leq \sup_{0 < t < T} \|v(t, \cdot)\|_{L^1(\Omega)}
\end{cases}$$

(11)

Now, let’s consider the following truncated system

$$\begin{cases}
    C_n \in W(H^1), \phi_n \in L^\infty(0, T, H^1_0(\Omega)) \\
\int_\Omega \frac{\partial C_{i,n}}{\partial t} + a^i_{\phi_n}(t, C_{i,n}, v) = \int_\Omega F^n_i(C_n)v + \lambda \int_\Omega C_{i,n}v \quad \forall v \in H^1(\Omega) \\
\varepsilon \int_\Omega \nabla \phi_n \nabla \psi = \int_\Omega \left(\sum_{i=1}^{N_s} z_i C_{i,n}^{(n)} \right) - f \right) \psi \quad \text{a.e } t > 0, \forall \psi \in H^1_0(\Omega) \\
\phi_n(0, x) = \phi_0(x) ; C_{i,n}(0, x) = C_{i,0}(x)
\end{cases}$$

(12)

where $W(H^1)$ is the Hilbert space defined by

$$W(H^1) = \left\{ u \in L^2(0, T, H^1(\Omega)^{N_s}); \frac{\partial u}{\partial t} \in L^2(0, T, H^{-1}(\Omega)^{N_s}) \right\}$$

let’s note that every $u \in W(H^1)$ is almost everywhere equal to a continuous function from $[0, T]$ in $L^2(\Omega)^{N_s}$. Furthermore,

$$W(H^1) \subset C([0, T] ; L^2(\Omega)^{N_s})$$

and the injection is continuous (see for more details Dautray and Lions [9]). So for $C_n \in W(H^1)$, the expression $C_{i,n}$ take for $t = 0$ the value $C_{i,0}$ which make sense, with an application $C_n \in W(H^1) \mapsto C_n(0) \in L^2(\Omega)^{N_s}$ continuous. Concerning our problem (12), we have the following result.

**Theorem 3.3** Under the hypothesis (6), (7) and (8) the problem (12) admits a weak solution $(C_n, \phi_n) \in W(H^1) \times L^\infty(0, T; W^{1,\infty}(\Omega))$ such that $C_n \geq 0$ in $Q_T$. 

Proof 3.4 Let’s consider $C_n = C_n e^{-\lambda t}$, then $C_n$ satisfies

\[
\begin{aligned}
\int_{\Omega} \frac{\partial C_{i,n}}{\partial t} \varphi + a_{\phi_n}^i (t, C_{i,n}, \varphi) &= \int_{\Omega} S^n_i(t, C_n) \varphi \quad \forall \varphi \in H^1(\Omega), \quad \text{with } S^n_i(t, C_n) = F^n_i(C_n e^{\lambda t}, e^{-\lambda t}) \\
\varepsilon \int_{\Omega} \nabla \phi_n \nabla \psi &= \int_{\Omega} \left( \sum_{i=1}^{N_s} z_i v_i^{(n)} e^{\lambda t} \right) \psi - f \quad \text{a.e } t > 0, \forall \psi \in H^1_0(\Omega) \\
\phi_n(0, x) &= \phi_0(x); C_{i,n}(0, x) = C_{i,0}(x)
\end{aligned}
\]

To prove the existence of a solution of (12), it suffices to prove the existence of a solution for the problem (13). We get this result by using the fixed point theorem of Schauder. For this we construct the following application:

\[
\mathcal{L}_n : L^2(Q_T)^{N_s} \rightarrow L^2(Q_T)^{N_s} \\
v \mapsto C_n
\]

where $\forall t \in [0, T], \phi_n$ is the unique solution of the elliptic problem

\[
\begin{aligned}
-\varepsilon \Delta \phi_n &= \sum_{i=1}^{N_s} z_i v_i^{(n)} e^{\lambda t} - f \quad \text{on } Q_T = \{ 0, T \times \Omega \} \\
\phi_n(t, x) &= 0 \quad \text{in } [0, T] \times \partial \Omega
\end{aligned}
\]

and $C_n = \mathcal{L}_n(v)$ satisfies the following parabolic system:

\[
\begin{aligned}
\int_{\Omega} \frac{\partial C_{i,n}}{\partial t} \varphi + a_{\phi_n}^i (t, C_{i,n}, \varphi) &= \int_{\Omega} S^n_i(t, v) \varphi \quad \forall \varphi \in H^1(\Omega) \\
C_{i,n}(0, x) &= C_{i,0}(x)
\end{aligned}
\]

In order to prove the last theorem, we use the Lax-Milgram theorem to prove that (15) has a unique solution (which prove that $\mathcal{L}_n$ is an application), then by using the fixed point theorem of Schauder, we prove that $\mathcal{L}_n$ admits a fixed point which will be the solution of (13), which ends this demonstration.

Concerning the solution of the problem (14), we make the following remark.

Remark 3.5 The solution of (14) satisfy

\[
\phi_n(t, x) = \int_0^t H(x,s) \theta^n_\varepsilon(t,s) ds
\]

where
\[
\theta^n_\varepsilon(t, s) = \frac{\sum_{i=1}^{N_s} \varepsilon v_i^{(n)}(t, s) e^{\lambda t}}{1 + \varepsilon \sum_{i=1}^{N_s} v_i^{(n)}(t, s) e^{\lambda t}} - f(s), \ s \in \Omega
\]

and \(H\) is the green function.

We have \(v^{(n)} \in C([0, T]; L^2(\Omega))\), and we have for every \(t \in [0, T], \phi_n(t, \cdot) \in H^2(\Omega)\) and \(\phi_n \in L^\infty(0, T; W^{1, \infty}(\Omega))\). Now, let’s make some assumptions on the bilinear form \(a^i_{\phi_n}\). We have the following result.

**Lemma 3.6** \(a^i_{\phi_n}\) is a continuous and coercive bilinear form on \(H^1(\Omega) \times H^1(\Omega)\), i.e. for every \(t \in [0, T]\)

1. There exists a positive constant \(C\) which depends only on \(d_i, m_i, \lambda, \|\nabla \phi_n\|_{L^\infty}\) such that

\[
\left| a^i_{\phi_n}(t, u, \tilde{u}) \right| \leq C \|u\|_{H^1(\Omega)} \|\tilde{u}\|_{H^1(\Omega)}
\]

2. There exists constants \(\alpha_0\) and \(\lambda^*\) strictly positive such that for every \(\lambda \geq \lambda^*\)

\[
a^i_{\phi_n}(t, u, u) \geq \alpha_0 \|u\|_{H^1(\Omega)}^2
\]

**Proof 3.7**

1. It’s obvious that \(a^i_{\phi_n}\) is a bilinear form on \(H^1(\Omega) \times H^1(\Omega)\), otherwise by using the Holder inequality and the fact that \(\phi_n \in L^\infty(0, T; W^{1, \infty}(\Omega))\), we have for every \((u, \tilde{u}) \in H^1(\Omega) \times H^1(\Omega)\)

\[
a^i_{\phi_n}(t, u, \tilde{u}) = \int_\Omega (d_i \nabla u \nabla \tilde{u} + m_i u \nabla \phi_n \nabla \tilde{u} + \lambda u \tilde{u})
\]

\[
\left| a^i_{\phi_n}(t, u, \tilde{u}) \right| \leq d_\infty \|u\|_{H^1(\Omega)} \|\tilde{u}\|_{H^1(\Omega)} + M \|\nabla \phi_n\|_{L^\infty} \|u\|_{H^1(\Omega)} \|\tilde{u}\|_{H^1(\Omega)} + \lambda \|u\|_{H^1(\Omega)} \|\tilde{u}\|_{H^1(\Omega)}
\]

where \(d_\infty = \max_{1 \leq i \leq N_s} (d_i)\), \(M = \max_{1 \leq i \leq N_s} (m_i)\), which prove that \(a^i_{\phi_n}\) is continuous on \(H^1(\Omega) \times H^1(\Omega)\).

2. Let’s put \(d_0 = \min_{1 \leq i \leq N_s} (d_i)\), we have

\[
a^i_{\phi_n}(t, u, u) \geq d_0 \|\nabla u\|_{L^2(\Omega)}^2 + m_i \int_\Omega u \nabla \phi_n \nabla u + \lambda \|u\|_{L^2(\Omega)}^2
\]
However, by using Holder’s and Young’s inequality, we got

\[ |m_i \int_\Omega u \nabla \phi_n \nabla u| \leq M \| \nabla \phi_n \|_{L^\infty} \| \nabla u \|_{L^2(\Omega)} \| u \|_{L^2(\Omega)} \]

\[ \leq \frac{\alpha}{2} \| \nabla u \|_{L^2(\Omega)}^2 + \frac{1}{2\alpha} (M \| \nabla \phi_n \|_{L^\infty})^2 \| u \|_{L^2(\Omega)}^2 \]

So

\[ a_{\phi_n}^i (t, u, u) \geq \left( d_0 - \frac{\alpha}{2} \right) \| \nabla u \|_{L^2(\Omega)}^2 + \left( \lambda - \frac{1}{2\alpha} (M \| \nabla \phi_n \|_{L^\infty})^2 \right) \| u \|_{L^2(\Omega)}^2 \]

Then, by choosing \( \alpha = d_0 \) and \( \lambda \geq \lambda^* = \frac{1}{2\alpha} (M \| \nabla \phi_n \|_{L^\infty})^2 \), we deduce that

\[ a_{\phi_n}^i (t, u, u) \geq \alpha_0 \| u \|_{H^1(\Omega)}^2 \text{ with } \alpha_0 = \min \left( \frac{d_0}{2}, \lambda - \lambda^* \right) \]

This implies that the bilinear form \( a_{\phi_n}^i \) is coercive on \( H^1(\Omega) \times H^1(\Omega) \).

Otherwise, the second member \( S_{n_i}^n (t, v) \) is fixed in \( L^2(Q_T) \), which allows us to apply the Lax-milgram theorem (see Dautray and Lions [9]), then we conclude for every \( v \in L^2(Q_T)^{Ns} \), the problem (15) admits a unique solution \( \overline{C}_n \).

Now, it’s time to prove that \( \mathcal{L}_n \) admits a fixed point. To this end, we prove though the following lemma, that the hypothesis for the Schauder fixed point theorem are satisfied. Thus, we have

**Lemma 3.8**  
1. \( \mathcal{L}_n \) is a continuous operator on \( L^2(Q_T)^{Ns} \).
2. \( \mathcal{L}_n \) sends \( L^2(Q_T)^{Ns} \) to the ball

\[
B = \left\{ v \in L^2(Q_T)^{Ns} \text{ such that } \| v \|_{L^2(Q_T)^{Ns}} \leq \sqrt{T \left( \frac{C_{n,T}}{2\alpha_0} + \| C_0 \|_{L^2(\Omega)}^2 \right)} \right\}
\]

where \( C_{n,T} \) is a constant depending on \( n \) and \( T \). Particularly \( \mathcal{L}_n (B) \subset B \).
3. \( \mathcal{L}_n \) is a compact operator.

**Proof 3.9**  
1. Let’s consider \( v \) and \( \overline{\varphi} \) in \( L^2(Q_T)^{Ns} \), \( \mathcal{L}_n (v) = \overline{C}_n \) and \( \mathcal{L}_n (\overline{\varphi}) = \hat{C}_n \).

We have for every \( \varphi \in H^1(\Omega) \)

\[
\int_\Omega \frac{\partial (\overline{C}_{i,n} - \hat{C}_{i,n})}{\partial t} \varphi + a_{\phi_n}^i (t, \overline{C}_{i,n} - \hat{C}_{i,n}, \varphi) = \int_\Omega (S^n_i (t, v) - \hat{S}_i^n (t, \hat{v})) \varphi
\]

we choose \( \varphi = \overline{C}_{i,n} - \hat{C}_{i,n} \), then we integrate on \([0, t]\) and by using the Young inequality and the coercivity of \( a_{\phi_n}^i \), we have
\[
\int_0^t \frac{\partial}{\partial t} \left\| \overline{C}_{i,n} - \overline{C}_{i,n} \right\|^2_{L^2(\Omega)} + 2\alpha_0 \left\| \overline{C}_{i,n} - \overline{C}_{i,n} \right\|^2_{H^1(\Omega)} \leq \int_0^t \left( \frac{1}{\alpha} \left\| S^n_i(t, v) - S^n_i(t, \tilde{v}) \right\|^2_{L^2(\Omega)} + \alpha \left\| \overline{C}_{i,n} - \overline{C}_{i,n} \right\|^2_{L^2(\Omega)} \right)
\]

By choosing \( \alpha = 2\alpha_0 \) and integrating on \([0, T]\)

\[
\left\| \overline{C}_{i,n} - \overline{C}_{i,n} \right\|_{L^2(Q_T)} \leq \sqrt{\frac{T}{2\alpha_0}} \left\| S^n_i(t, v) - S^n_i(t, \tilde{v}) \right\|_{L^2(Q_T)}
\]

By using the fact that \( S^n_i \) is globally lipschitz, which enable us to deduce the existence of a constant \( C_{n,T} \) depending only on \( n \) and \( T \) such that

\[
\left\| S^n_i(t, v) - S^n_i(t, \tilde{v}) \right\|_{L^2(Q_T)} \leq C_{n,T} \left\| v - \tilde{v} \right\|_{L^2(Q_T)}^{N_s}
\]

Consequently

\[
\left\| \mathcal{L}_n(v) - \mathcal{L}_n(\tilde{v}) \right\|_{L^2(Q_T)^{N_s}} = \left\| \overline{C}_{i,n} - \overline{C}_{i,n} \right\|_{L^2(Q_T)^{N_s}} \leq C_{n,T} \sqrt{\frac{T}{2\alpha_0}} \left\| v - \tilde{v} \right\|_{L^2(Q_T)^{N_s}}
\]

2. We are going to prove that \( \mathcal{L}_n \left( L^2(Q_T)^{N_s} \right) \subset B \). Let’s consider \( v \) in \( L^2(Q_T)^{N_s} \) and \( \mathcal{L}_n(v) = \overline{C}_n \), we have

\[
\int_{\Omega} \frac{\partial}{\partial t} \varphi + a^i_{\phi_n}(t, \overline{C}_{i,n}, \varphi) = \int_{\Omega} S^n_i(t, \overline{C}_n) \varphi
\]

we take \( \varphi = \overline{C}_{i,n} \), we have just like the last proof

\[
\left\| \overline{C}_{i,n}(t) \right\|^2_{L^2(\Omega)} + 2\alpha_0 \int_0^t \left\| \overline{C}_{i,n} \right\|^2_{H^1(\Omega)} \leq \int_0^t \left( \frac{1}{\alpha} \left\| S^n_i(t, v) \right\|^2_{L^2(\Omega)} + \alpha \left\| \overline{C}_{i,n} \right\|^2_{H^1(\Omega)} + \left\| C_{i,0} \right\|^2_{L^2(\Omega)} \right)
\]

we choose \( \alpha = 2\alpha_0 \) to obtain

\[
\left\| \overline{C}_{i,n}(t) \right\|^2_{L^2(\Omega)} \leq \frac{1}{2\alpha_0} \left\| S^n_i(t, v) \right\|^2_{L^2(\Omega)} + \left\| C_{i,0} \right\|^2_{L^2(\Omega)}
\]

\[
\cdots \leq \frac{C_{n,T}}{2\alpha_0} + \left\| C_{i,0} \right\|^2_{L^2(\Omega)}
\]

We integrate on \([0, T]\), and we conclude that

\[
\left\| \mathcal{L}_n(v) \right\|_{L^2(Q_T)^{N_s}} \leq \sqrt{T \left( \frac{C_{n,T}}{2\alpha_0} + \left\| C_{i,0} \right\|^2_{L^2(\Omega)^{N_s}} \right)}
\]
3. Let $B$ a bounded domain of $L^2(Q_T)^{Ns}$, we need to prove that $\mathcal{L}_n(B)$ is bounded in $W(H^1)$. Let be $v^k \in B \subset L^2(Q_T)^{Ns}$ and $\mathcal{L}_n(v^k) = \overline{C}_n^k$. We have according to (16)

$$\left\| \overline{C}_{i,n}(t) \right\|_{L^2(\Omega)}^2 + 2\alpha_0 \int_0^T \left\| \overline{C}_{i,n} \right\|_{H^1(\Omega)}^2 \leq \int_0^T \frac{1}{\alpha} \left\| S_i^n(t, v^k) \right\|_{L^2(\Omega)}^2 + \alpha \int_0^T \left\| \overline{C}_{i,n} \right\|_{H^1(\Omega)}^2 + \left\| C_{i,0}^k \right\|_{L^2(\Omega)}^2$$

Let’s take $\alpha = \alpha_0$,

$$\alpha_0 \int_0^T \left\| \overline{C}_{i,n} \right\|_{H^1(\Omega)}^2 \leq T \left( \frac{C_{n,T}}{\alpha_0} + \left\| C_{i,0}^k \right\|_{L^2(\Omega)} \right) \leq R$$

and this implies that the subsequence $\mathcal{L}_n(v^k) = \overline{C}_n$ is bounded in $L^2(0, T; H^1(\Omega)^{Ns})$. Otherwise $\overline{C}_{i,n}$ satisfies

$$\int_\Omega \frac{\partial \overline{C}_{i,n}^k}{\partial t} \phi + a_{in}^k(t, \overline{C}_{i,n}^k, \phi) = \int_\Omega S_i^n(t, v^k) \phi$$

we have

$$\left\| \int_\Omega \frac{\partial \overline{C}_{i,n}^k}{\partial t} \phi \right\| \leq C \left\| \overline{C}_{i,n}^k \right\|_{H^1(\Omega)} \left\| \phi \right\|_{H^1(\Omega)} + \left\| S_i^n(t, v^k) \right\|_{L^2(\Omega)} \left\| \phi \right\|_{L^2(\Omega)}$$

$$\ldots \leq C \left\| \overline{C}_{i,n}^k \right\|_{H^1(\Omega)} \left\| \phi \right\|_{H^1(\Omega)} + C_{n,T} \left\| \phi \right\|_{H^1(\Omega)}$$

Then,

$$\int_0^T \left\| \frac{\partial \overline{C}_{i,n}^k}{\partial t} \right\|_{H^{-1}(\Omega)}^2 \leq 2 \int_0^T \left( C \left\| \overline{C}_{i,n}^k \right\|_{H^1(\Omega)}^2 + C_{n,T} \right)$$

This allows us to deduce $\frac{\partial \overline{C}_{i,n}^k}{\partial t}$ is bounded in $L^2(0, T; H^{-1}(\Omega)^{Ns})$. We know that we have the injections, $L^2(0, T; H^1(\Omega)^{Ns}) \subset L^2(Q_T)^{Ns}$ and $L^2(0, T; H^{-1}(\Omega)^{Ns}) \subset L^2(Q_T)^{Ns}$ are compact, consequently we conclude that $\mathcal{L}_n$ is a compact operator.

Finally, the operator $\mathcal{L}_n$ admits a fixed point $C_n$ such that $(C_n, \phi_n)$ is the solution we are seeking. Now, we have to prove the positivity of $C_n$, for that we introduce the function $Z_n = (Z_{i,n})_{1 \leq i \leq Ns}$ defined by

$$Z_{i,n} = C_{i,n} \exp \left( \frac{t}{\delta_i} (\phi_n) \right) \text{ for } i = 1, 2, \ldots, Ns.$$ 

Moreover, we consider


\[ p_{i,n} = \exp\left( \frac{m_i}{d_i} (\phi_n) \right) \] and \[ q_{i,n} = \frac{1}{p_{i,n}} \]

for every \( v \in C^1(Q_T) \), \( Z_{i,n} \) satisfies

\[
\begin{cases}
  \int_0^t \int_\Omega \frac{\partial (q_{i,n} Z_{i,n})}{\partial t} v + d_i \int_0^t \int_\Omega q_{i,n} \nabla Z_{i,n} \nabla v = \int_0^t \int_\Omega F^n_i (q_n Z_n) v \\
  Z_{i,n}(0,x) = C_{i,0}(x)p_{i,n}(0,x)
\end{cases}
\] \quad \forall x \in \Omega \quad (17)

Let’s introduce the function \( \text{sign}^- \) defined on \( IR \) by

\[ \text{sign}^- r = \begin{cases} 
  -1 & \text{if } r < 0 \\
  0 & \text{if } r \geq 0
\end{cases} \]

as \( \text{sign}^- \) is increasing function, we consider the convex function \( j_\varepsilon \in C^2(IR) \) such that

\[ j_\varepsilon'(r) \rightarrow \text{sign}^- r \quad \text{when} \quad \varepsilon \rightarrow 0 \]

Let’s take \( j_\varepsilon'(Z_{i,n}) \) as a test function in (17), we have then

\[
\int_0^t \int_\Omega \frac{\partial (q_{i,n} Z_{i,n})}{\partial t} j_\varepsilon'(Z_{i,n}) = -d_i \int_0^t \int_\Omega q_{i,n} \nabla Z_{i,n} \nabla (j_\varepsilon'(Z_{i,n})) + \int_0^t \int_\Omega F^n_i (q_n Z_n) j_\varepsilon'(Z_{i,n}) \quad (18)
\]

Let’s note by \( I_1 \) and \( I_2 \), respectively the two members in the right side of the equality (18). Using the convexity of \( j_\varepsilon \), we deduce for the first integral

\[ I_1 = -d_i \int_0^t \int_\Omega q_{i,n} |\nabla Z_{i,n}|^2 j_\varepsilon''(Z_{i,n}) \leq 0 \]

Concerning the second member \( I_2 \), we have

\[
\lim_{\varepsilon \rightarrow 0} I_2 = \lim_{\varepsilon \rightarrow 0} \int_0^t \int_\Omega F^n_i (q_n Z_n) j_\varepsilon'(Z_{i,n}) = \lim_{\varepsilon \rightarrow 0} \int_{Z_{i,n} \geq 0} F^n_i (q_n Z_n) j_\varepsilon'(Z_{i,n}) + \lim_{\varepsilon \rightarrow 0} \int_{Z_{i,n} < 0} F^n_i (q_n Z_n) j_\varepsilon'(Z_{i,n}) \]

\[ = \lim_{\varepsilon \rightarrow 0} \int_{Z_{i,n} < 0} F^n_i (q_n Z_n) j_\varepsilon'(Z_{i,n}) \]

by (10) we have

\[
\lim_{\varepsilon \rightarrow 0} I_2 = - \int_{Z_{i,n} < 0} F^n_i (q_n Z_n) = - \int_{Z_{i,n} < 0} \eta_n(|q_n Z_n|) \hat{F}_i(q_n Z_n) = - \int_{Z_{i,n} < 0} \eta_n(|q_n Z_n|) F_i((q_n Z_n)^+) \]

by (6)
\[
\lim_{\varepsilon \to 0} I_2 \leq 0
\]

Then
\[
\lim_{\varepsilon \to 0} \int_0^t \int_\Omega \frac{\partial (q_{i,n} Z_{i,n})}{\partial t} j'_\varepsilon (Z_{i,n}) \leq 0
\]

which means
\[
\int_0^t \int_\Omega \frac{\partial (q_{i,n} Z_{i,n})}{\partial t} \text{sign}^- (Z_{i,n}) \leq 0
\]

consequently
\[
\int_0^t \int_\Omega \frac{\partial (q_{i,n} Z_{i,n})^-}{\partial t} \leq 0
\]

which implies
\[
\int_\Omega (q_{i,n} Z_{i,n})^- (t, x) \leq \int_\Omega (q_{i,n} Z_{i,n})^- (0, x)
\]

as \((q_{i,n} Z_{i,n})(0, x) \geq 0\) almost for every \(x \in \Omega\), we deduce that
\[
\int_\Omega (q_{i,n} Z_{i,n})^- (t, x) \leq 0
\]

Finally \((q_{i,n} Z_{i,n})^- = 0\) and then \(Z_{i,n} \geq 0\), which allows us to conclude that \(C_{i,n} \geq 0\) for every \(i = 1, 2, \ldots, N_s\).

### 3.3.2 A priori estimates

Till the end of this chapter we design by \(C\) every generic and positive constant. This constant can change its value in different situations, but remains independent of \(n\). In this paragraph we give estimations concerning \(C_n, \phi_n\) and \(F_n(C_n)\) in appropriate spaces. We start by proving in the following lemma, that \(\sup_{0 < t < T} \|C_n(t)\|_{L^1(\Omega)^{N_s}}\) and \(\|\phi_n\|_{L^\infty}\) are bounded independently of \(n\).

**Lemma 3.10** There exist a constant \(C\) depending only on \(T\) and \(\|C_0\|_{L^2(\Omega)^{N_s}}\) such that

1. \(\sup_{0 < t < T} \int_\Omega \sum_{i=1}^{N_s} |C_{i,n}(t, x)| \, dx \leq C\)

2. \(\|\phi_n\|_{L^\infty(0,T;W^{1,\infty}(\Omega))} \leq C\)
Proof 3.11  1. Let’s remember that \((C_n, \phi_n)\) solution of our approximated problem satisfies:

\[
\int_{\Omega} \frac{\partial C_{i,n}^n}{\partial t} \varphi + d_i \int_{\Omega} \nabla C_{i,n} \nabla \varphi + \int_{\Omega} m_i C_{i,n} \nabla \phi_n \nabla \varphi = \int_{Q_T} F_i^n(C_n) \varphi \quad (19)
\]

we take \(\varphi = 1\), and we multiply the equation (19) by \(\alpha_i\)

\[
\alpha_i = \begin{cases} 
1 & \text{if } i \in A_p \\
\frac{\text{card}(A_p)}{\text{card}(A_s)} & \text{if } i \in A_s
\end{cases}
\]

then we sum on \(i\)

\[
\sum_{i=1}^{N_s} \alpha_i F_i^n(C_n) = 0
\]

this implies

\[
\sum_{i=1}^{N_s} \alpha_i \int_{\Omega} \frac{\partial C_{i,n}^n}{\partial t} = 0
\]

we integrate the last equality on \([0, t]\), for every \(0 < t < T\) we deduce

\[
\sum_{i=1}^{N_s} \alpha_i \int_{\Omega} C_{i,n} (t, x) \, dx = \sum_{i=1}^{N_s} \alpha_i \int_{\Omega} C_{i,0} (x) \, dx
\]

furthermore,

\[
\sum_{i=1}^{N_s} \int_{\Omega} |C_{i,n} (t, x)| \, dx \leq C \left( \sum_{i=1}^{N_s} \|C_{i,0}\|_{L^2(\Omega)} \right).
\]

Consequently,

\[
\sup_{0 < t < T} \int_{\Omega} \sum_{i=1}^{N_s} |C_{i,n} (t, x)| \, dx \leq C
\]

2. Let’s remember that \(\phi_n\) satisfy the equation

\[
\begin{cases}
- \varepsilon \Delta \phi_n (t, x) = \frac{\sum_{i=1}^{N_s} z_i C_{i,n}^{(n)}(t, x) e^{\lambda t}}{1 + \varepsilon \sum_{i=1}^{N_s} C_{i,n}^{(n)}(t, x) e^{\lambda t}} - f(x) & \text{in } Q_T \\
\phi_n(t, x) = 0 & \text{in } ]0, T[ \times \partial \Omega
\end{cases}
\]

then

\[
\phi_n (t, x) = \int_{\Omega} H(x, s) \left( \frac{\sum_{i=1}^{N_s} z_i C_{i,n}^{(n)}(t, s) e^{\lambda t}}{1 + \varepsilon \sum_{i=1}^{N_s} C_{i,n}^{(n)}(t, s) e^{\lambda t}} - f(s) \right) ds
\]
we have
\[\begin{align*}
\left\| \sum_{i=1}^{N_s} z_i C_{i,n} e^{\lambda t} \right\|_{L^\infty(Q_T)} - f & \leq C \\
1 + \varepsilon \sum_{i=1}^{N_s} C_{i,n} e^{\lambda t} & \leq C
\end{align*}\]

then we have
\[\|\phi_n\|_{L^\infty(0,T;W^{1,\infty}(\Omega))} \leq C\]

Concerning the term \(F^n_i(C_n)\), we have the following result.

**Lemma 3.12** There exist a constant independent on \(n\), such that for \(i = 1, 2, \ldots, N_s\)
\[\int_0^T \int_\Omega |F^n_i(C_n)| \leq C\]

**Proof 3.13** Let’s consider the equation satisfied by \(C_{i,n}\) for some \(i \in A_s\) we have
\[\frac{\partial C_{i,n}}{\partial t} - d_i \Delta C_{i,n} - m_i \text{div}(C_{i,n}\nabla \phi) = -F^n_i(C_n)\]

Then we integrate on \(Q_T\)
\[\int_{Q_T} F^n_i(C_n) = - \int_{Q_T} \frac{\partial C_{i,n}}{\partial t} = \int_\Omega C_{i,0}(x) - \int_\Omega C_{i,n}(T,x) \leq \int_\Omega C_{i,0}(x)\]
because we know that \(C_{i,n}(T,x) \geq 0\)
we conclude that
\[\int_{Q_T} F^n_i(C_n) \leq C\]

Finally, to have the assumptions on \(F^n_i\) for \(i \in A_p\), we use the fact that
\(|F^n_i| = F^n_i(C_n)|\), to deduce that
\[\int_0^T \int_\Omega |F^n_i(C_n)| \leq C\]

For \(i \in A_0\) the result is obvious.

The following lemma gives estimations on \(C_{i,n}\) in \(L^2(0,T;H^1(\Omega))\), and \(C_{i,n}F^n_i(C_n)\) in \(L^1(Q_T)\). Those estimations will be very important to fulfill the proof of the main result.

**Lemma 3.14** For \(i = 1, \ldots, N_s\), there exists a constant \(C\) depending only on \(d_i, m_i\) and \(\|C_{i,0}\|_{L^2(\Omega)}\) such that:
1. \( \|C_{i,n}\|_{L^2(0,T;H^1(\Omega))} \leq C \)

2. \( \|C_{i,n}F^n_i(C_n)\|_{L^1(Q_T)} \leq C \)

**Proof 3.15** We start by proving the lemma for some \( i \in A_k \). We multiply in this case the first equation in (12) by \( C_{i,n} \). Then by integrating on \( Q_t \) we have

\[
\frac{1}{2} \int_\Omega \left( C_{i,n}^2(t,x) - C_{i,0}^2(x) \right) + d_i \int_{Q_t} |\nabla C_{i,n}|^2 + m_i \int_{Q_t} C_{i,n} \phi_n \nabla C_{i,n} + \int_{Q_t} C_{i,n} |F^n_i(C_n)| = 0
\]

then

\[
\frac{1}{2} \int_\Omega C_{i,n}^2(t,x) + d_i \int_{Q_t} |\nabla C_{i,n}|^2 \leq -m_i \int_{Q_t} C_{i,n} \phi_n \nabla C_{i,n} + \frac{1}{2} \int_\Omega C_{i,0}^2(x)
\]

as \( \|\nabla \phi_n\|_{L^\infty(Q_T)} \leq C \), and by the Young inequality

\[
\frac{1}{2} \int_\Omega C_{i,n}^2(t,x) + d_i \int_{Q_t} |\nabla C_{i,n}|^2 \leq \varepsilon \int_{Q_t} |\nabla C_{i,n}|^2 + C_\varepsilon \int_{Q_t} C_{i,n}^2 + \frac{1}{2} \int_\Omega C_{i,0}^2(x)
\]

then

\[
\frac{1}{2} \int_\Omega C_{i,n}^2(t,x) + (d_i - \varepsilon) \int_{Q_t} |\nabla C_{i,n}|^2 \leq C_\varepsilon \int_{Q_t} C_{i,n}^2 + \frac{1}{2} \int_\Omega C_{i,0}^2(x)
\]

Consequently

\[
\|C_{i,n}\|_{L^2(0,T;H^1(\Omega))} \leq C
\]

Otherwise we have

\[
\int_{Q_T} C_{i,n} |F^n_i(C_n)| \leq -m_i \int_{Q_T} C_{i,n} \phi_n \nabla C_{i,n} + \frac{1}{2} \int_\Omega C_{i,0}^2(x)
\]

we use the same reasons to prove that

\[
\|C_{i,n}F^n_i(C_n)\|_{L^1(Q_T)} \leq C
\]

The case for \( i \in A_0 \) is obvious by using the same last arguments. Now let’s go to the cases for \( j \in A_p \). To this end we consider the equation satisfied by \( W_n = C_{j,n} + C_{i,n} \) and we use the fact that \( F^n_i = -F^n_j \). We have then

\[
\frac{\partial W_n}{\partial t} - d_j \Delta W_n + (d_j - d_i) \Delta C_{i,n} - m_j \text{div}(W_n \nabla \phi_n) + (m_j - m_i) \text{div}(C_{i,n} \nabla \phi_n) = 0
\]

we multiply the last equation by \( W_n \) and we integrate on \( Q_t \). By following the same steps as ago, we have

\[
\frac{1}{2} \int_{Q_t} \frac{\partial}{\partial t} W_n^2 + d_j \int_{Q_t} |\nabla W_n|^2 \leq (d_j - d_i) \int_{Q_t} \nabla C_{i,n} \nabla W_n + \int_{Q_t} (m_i C_{i,n} + m_j C_{j,n}) \nabla \phi_n \nabla W_n
\]
then
\[
\frac{1}{2} \int_{\Omega} \frac{\partial}{\partial t} W_n^2(t, x) + d_j \int_{Q_t} |\nabla W_n|^2 \leq \frac{1}{2} \int_{\Omega} \frac{\partial}{\partial t} W_n^2(0, x) + (d_j - d_i) \int_{Q_t} \nabla C_{i,n} \nabla W_n + \int_{Q_t} (m_i C_{i,n} + m_j C_{j,n}) \nabla \phi_n \nabla W_n
\]

By using the Young and Holder inequality, we got
\[
\frac{1}{2} \int_{\Omega} W_n^2(t, x) + d_j \int_{Q_t} |\nabla W_n|^2 \leq \frac{1}{2} \int_{\Omega} \left( C_{j,0}^2 + C_{i,0}^2 \right) (0, x) + \varepsilon_1 \int_{Q_t} |\nabla W_n|^2 + C_{\varepsilon_1} \int_{Q_t} |\nabla C_{i,n}|^2
\]
\[
+ \varepsilon_2 \int_{Q_t} |\nabla W_n|^2 + C_{\varepsilon_2} \int_{Q_t} |m_i C_{i,n} + m_j C_{j,n}|^2
\]

which ends the prove. Concerning the term on \( C_{j,n} F_j^n (C_n) \) for \( j \in A_p \), we have
\[
\int_{Q_T} C_{j,n} F_j^n (C_n) \leq \frac{1}{2} \int_{\Omega} \left( C_{j,n}^2 (t, x) - C_{j,0}^2 (x) \right) + d_j \int_{Q_T} |\nabla C_{j,n}|^2 + m_j \int_{Q_T} C_{j,n} \nabla \phi_n \nabla C_{i,n}
\]
using the fact that \( \| C_{j,n} \|_{L^2(0,T;H^1(\Omega))} \leq C \) and \( \| \nabla \phi_n \|_{L^\infty(Q_T)} \leq C \), then we deduce the result we are looking for.

Those estimations will be very useful for passing to the limit in the approximated problem (12).

### 3.3.3 Convergence

The point is to show that \((C_n, \phi_n)\) solution of the problem (12) converge to \((C, \phi)\) solution of (5). By lemma 3.12 and lemma 3.14, and by using the result of Barass, Hassan and Veron [8], we can deduce the existence of a subsequence denoted \((C_{i,n}, \phi_{i,n})\), such that for all \( i = 1, 2, ..., N_s \)

\[
\begin{align*}
C_{i,n} & \rightarrow C_i \text{ strongly in } L^1(Q_T) \\
C_{i,n} & \rightarrow C_i \text{ almost everywhere in } Q_T
\end{align*}
\]

as \( \phi_n \) is uniformly bounded in \( L^\infty (0,T;W^{1,\infty} (\Omega)) \), we conclude the existence of \( \phi \in L^\infty (0,T;W^{1,\infty} (\Omega)) \) such that :

\[
\nabla \phi_n \rightarrow \nabla \phi \text{ for the topology } \sigma \left( L^\infty (Q_T), L^1 (Q_T) \right)
\]

then, let’s prove that

\[
C_{i,n} \nabla \phi_n \rightarrow C_i \nabla \phi \text{ in } D'(Q_T)
\]

To this end we need to prove that

\[
C_{i,n} \nabla \phi_n \rightarrow C_i \nabla \phi \text{ for the topology } \sigma \left( L^1(Q_T), L^\infty(Q_T) \right)
\]
Let be $\Psi \in L^\infty (Q_T)$, we have

$$
\int_0^T \int_\Omega \Psi \left( C_{i,n} \nabla \phi_n - C_i \nabla \phi \right) dx dt = \int_0^T \int_\Omega \Psi \nabla \phi_n \left( C_{i,n} - C_i \right) dx dt
$$

$$
+ \int_0^T \int_\Omega \Psi C_i \left( \nabla \phi_n - \nabla \phi \right) dx dt
$$

concerning the first term of the last equality, we have

$$
\left| \int_0^T \int_\Omega \Psi \nabla \phi_n \left( C_{i,n} - C_i \right) dx dt \right| \leq \| \Psi \|_{L^\infty} \| \nabla \phi_n \|_{L^\infty} \| C_{i,n} - C_i \|_{L^1}
$$

as

$$
\| C_{i,n} - C_i \|_{L^1} \to 0
$$

we deduce that

$$
\int_0^T \int_\Omega \Psi \nabla \phi_n \left( C_{i,n} - C_i \right) dx dt \to 0
$$

The second term goes to zero too because $\Psi C_i \in L^1 (Q_T)$ and $\nabla \phi_n$ converge to $\nabla \phi$ for the topology $\sigma (L^\infty (Q_T), L^1 (Q_T))$. Consequently,

$$
\frac{\partial C_{i,n}}{\partial t} - d_i \Delta C_{i,n} - m_i \text{div}(C_{i,n} \nabla \phi_n)
$$

converge in $D' (Q_T)$ to

$$
\frac{\partial C_i}{\partial t} - d_i \Delta C_i - m_i \text{div}(C_i \nabla \phi)
$$

Otherwise, we know that for every $t \in ]0, T[

$$
\phi_n (t, x) = \int_\Omega H (x, s) \theta^n (t, s) ds
$$

where

$$
\theta^n (t, x) = \frac{\sum_{i=1}^{N_s} z_i C_{i,n}^{(n)} (t, x)}{1 + \varepsilon \sum_{i=1}^{N_s} C_{i,n}^{(n)} (t, x)} - f (x)
$$

by using (20), (21) and (11) we have
As $F_i$ is continuous and $F_i^n$ is the truncation of $F_i$, we have furthermore

$$F_i^n(C_n) \rightarrow F_i(C) \text{ almost everywhere in } Q_T$$

This is not enough to pass to the limit in (12) and conclude that $(C, \phi)$ is a solution of (5), indeed we must prove that the convergence (22) is in $L^1(Q_T)$.

Thanks to Vitali theorem, to prove that $F_i^n(C_n)$ converge to $F_i(C)$ in $L^1(Q_T)$ is equivalent to prove that $F_i^n(C_n)$ is equi-integrable in $L^1(Q_T)$. We have the following lemma:

\textbf{Lemma 3.16} for every $i = 1, 2, \ldots, N_s$, $F_i^n(C_n)$ is equi-integrable in $L^1(Q_T)$.

\textbf{Proof 3.17} Let be $E$ a measurable set of $Q_T$. We have

$$\int_E |F_i^n(C_n)| \leq \int_{E \cap [C_{i,n} \leq k]} |F_i^n(C_n)| + \frac{1}{k} \int_{E \cap [C_{i,n} > k]} C_{i,n} |F_i^n(C_n)|$$

However

$$\int_{E \cap [C_{i,n} \leq k]} |F_i^n(C_n)| \leq \max_{0 \leq |r| \leq k} |F(r)| \cdot |E|$$

$$\ldots \leq C(k) |E|$$

according to lemma 3.14

$$\frac{1}{k} \int_{E \cap [C_{i,n} > k]} C_{i,n} |F_i^n(C_n)| \leq \frac{C(T)}{k}$$

by choosing $k$ sufficiently large, we deduce

$$\int_{E \cap [C_{i,n} \leq k]} |F_i^n(C_n)| \leq \frac{\varepsilon}{2} \text{ and } \frac{1}{k} \int_{E \cap [C_{i,n} > k]} C_{i,n} |F_i^n(C_n)| \leq \frac{\varepsilon}{2}$$

consequently, $F_i^n(C_n)$ is equi-integrable in $L^1(Q_T)$, which ends the proof of the theorem 3.2.
References


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