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A Ranking Method in Dynamic DEA

Maryam Reshadi[a]

Department of Mathematics, East Tehran Branch
Islamic Azad University, Tehran, Iran

Mohammad Reza Moazami Goudarzi

Department of Mathematics, Borujerd Branch
Islamic Azad University, Borujerd, Iran

Abstract

Dynamic DEA is an approach by which the performance of units is assessed through a few non-separate time periods covered by a panel named assessment window. This paper firstly defines link factors which bear connectivity between those time periods. Then, a new version of dynamic DEA is developed for which units could be ranked after a while assessment at the of the assessment window.

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1 Introduction

In the late of 1970's DEA was originated by Charnes, Cooper and Rhodes (CCR) [1] as a method to evaluate relative efficiency measure of DMUs. Afterwards, dynamic DEA (DDEA, hereafter) was originally developed by Färe et al. [2] to cope with a long time assessment incorporating concepts of quasi-fixed inputs or investment activities. In DDEA, the performance of a DMU is studied not only in an assessment period, but also continues over a few assessment periods covered by a panel named an assessment window. In fact an assessment window plays a key role in related to DDEA framework. Actually, what discriminates DDEA models from the other time dependent models like Window analysis [3] and the Malmquist indices [4,5,6] is the existence of link

Corresponding author. e-mail: Maryam_Reshadi@yahoo.com

factors such as intermediate outputs, quasi-fixed inputs or investment activities that all of them yield connectivity between assessment periods. Therefore, the DDEA yields some worthy results to DMs by which they can judge truly about DMUs'performance. In DDEA literature many researches, e.g. Nemoto and Goto [7,8], Sueyoshi and Sekitani [9] developed various subjects such as dynamic returns to scale (DRTS), dynamic overall efficiency (DOE), dynamic slack-based measure (DSBM), etc. But none of them have noted to the ranking issue in DDDEA context. This paper firstly develops a new version of DDEA in which the links factors will be mentioned accurately. Then into this new DDEA context, we propose one method to rank DMUs at the end of the assessment window. The rest of the paper is as follows: Section 2 introduces some preliminaries of the basic DDEA models. Section 3 develops some link factors and consequently new version of DDEA to propose one ranking index. Finally, section 4 shows some concluding remarks.

2 Preliminary of DDEA

This section exhibits some definitions into the basic DDEA. Assume that there are n DMUs which are assessed in T non-separate assessment periods belonging to an assessment window, namely W. Nemoto and Goto's (hereafter NG) defined ϕ_t^{CRS} as a dynamic production possibility set as follows:

$$\phi_t^{CRS} = \left\{ (x_t, k_{t-1}, y_t, k_t) \in \mathbb{R}_+^{m+l} \times \mathbb{R}_+^{s+l} | (x_t, k_{t-1}) \text{ can produce } (y_t, k_t) \right\}. \tag{1}$$

Suppose $\lambda_t = (\lambda_{t,1}, \lambda_{t,2}, ..., \lambda_{t,n})^\tau$ as a weight vector in which τ represents the transpose of a vector, and $X_t = [x_{t,1}, x_{t,2}, ..., x_{t,n}]$, $K_{t-1} = [k_{t-1,1}, k_{t-1,2}, ..., k_{t-1,n}]$ and $Y_t = [y_{t,1}, y_{t,2}, ...y_{t,n}]$ as the matrices of inputs, quasi-fixed inputs and outputs, respectively. Then, the above-mentioned ϕ_t^{CRS} can be rewritten as follows:

$$\phi_t^{CRS} = \tag{2}$$
$$\left\{ (x_t, k_{t-1}, y_t, k_t) \in \mathbb{R}_+^{m+l} \times \mathbb{R}_+^{s+l} | X_t\lambda_t \leq x_t, K_{t-1}\lambda_t \leq k_{t-1}, Y_t\lambda_t \geq y_t, K_t\lambda_t \geq k_t, \lambda_t \geq \mathbf{0} \right\}.$$

The "$\mathbf{0}$" is used as a vector whose all components are 0. Indeed, in the tth period of assessment (t=1,...,T), a DMU_p utilizes an input vector $(x_t, k_{t-1}) \in \mathbb{R}_+^{m+l}$ to yield an output vector $(y_t, k_t) \in \mathbb{R}_+^{s+l}$. Note that k_t produced in the tth period of assessment might be used as a vector of quasi-fixed inputs in the $t+1$th period of assessment. Here, $W = \{1, 2, ..., T\}$ shows an assessment window whose assessment periods are non-separate. That means several interconnected factors or link factors such as quasi-fixed inputs, intermediate outputs or investment activities might be between its assessment

periods. Here, some link factors are defined as follows.
1. Quasi-fixed inputs: quasi-fixed or non-discretionary inputs are those that may not be restricted, such as acres of land in a farm.
2. Investment inputs: special inputs that could be referred as constrained discretionary variables that those constraints may come from a specific investment policy.

3 Main Results

3.1 The link factors in DDEA

Suppose that $\mathbf{X}_{t,p}$ is a vector of whole inputs consumed by a DMU_p at the beginning of the time period t that $\mathbf{X}_{t,p} = \begin{pmatrix} x_{t,p} \\ A_{t,p}k_{t-1,p} \\ \overline{x}_p \end{pmatrix}$; where $x_{t,p} = (x_{t,p,1}, ..., x_{t,p,m})$ is a m-vector of inputs whose the components might be different from one assessment period to another one; $k_{t-1,p} = (k_{t-1,p,1}, ..., k_{t-1,p,\overline{s}})$ is a $\overline{s}$-vector of the investment inputs which come from the assessment period t-1 to use in the assessment period t; A_t is a $s \times s$-diagonal matrix whose the ith diagonal element a_{ii} $(i = 1, ..., s)$ is a usage percent of the ith component of the $k_{t-1,p}$, i.e. $k_{t-1,p,i}$. Therefore, $A_{t,p}k_{t-1,p}$ is a share of the investment inputs vector $k_{t-1,p}$ that is used in the assessment period period t. The rest of the $k_{t-1,p}$ is not used and is sent to the assessment period t+1. That means if $\widetilde{A}_{t,p}$ to be a $s \times s$-diagonal matrix whose the ith diagonal element is $1 - a_{ii}$, $(i = 1, ..., s)$ as a non-usage percent of the $k_{t-1,p,i}$, then the rest of the investment inputs which are not used in the assessment period t and are sent to the assessment period t+1 could be shown as $\widetilde{A}_{t,p}k_{t-1,p}$. Further, suppose that $\mathbf{Y}_{t,p} \in \mathbb{R}_+^{s+l}$ to be as a vector of all outputs produced by the DMU_p in the assessment period t where $\mathbf{Y}_{t,p} = \begin{pmatrix} y_{t,p} \\ h_{t,p} \end{pmatrix}$. The $y_{t,p} \in \mathbb{R}_+^s$ is a vector of external outputs that are sent to markets and $h_{t,p} \in \mathbb{R}_+^l$ is a vector of internal outputs that are sent to the assessment period t+1. Moreover, DMU_p might consume a vector of quasi-fixed inputs shown by $\overline{x} \in R_+^{\overline{m}}$, whose components will be same to all DMUs and all assessment periods such as acres of a land in a farm. To clarify the connectivity between two assessment periods in DDEA framework the following constraint should be hold to the link factors:

$$h_t + \widetilde{A}_t k_{t-1} = k_t = A_{t+1}k_t + \widetilde{A}_{t+1}k_t, \qquad t = 1, ..., T-1. \tag{3}$$

Fig. The link factors between two assessment periods t and t+1.

By considering the above figure, Eq. (7) could be summarized as follows:

$$h_t + \widetilde{A}_t k_{t-1} = A_{t+1}k_t + \widetilde{A}_{t+1}k_t, \qquad t = 1, ..., T-1. \tag{4}$$

3.2 Ranking index

In this section firstly a frontier called ideal frontier (IF, hereafter) is developed. Assume that W shows the assessment window containing T non-separate time periods, $W = \{1, 2, ..., T\}$. The frontier IF is made up of a set of virtual DMUs called ideal DMUs and collected into the set IS^W as follows:

$$IS^W = \left\{DMU_I^t, t \in W\right\}, \tag{5}$$

Where DMU_I^t is an ideal DMU in the time period t that $DMU_{I,t} = (X_{I,t}, Y_{I,t})$. The $X_{I,t}$ is defined as follows: $\mathbf{X}_{I,t} = \begin{pmatrix} I_i^{t.var-int} = & \min_j x_{i,t,j}, \ i = 1, ..., m \\ I_i^{t.cap-int} = & \min_j A_{t,j} k_{i,t-1,j}, \ i = 1, ..., s \\ I_i^{qua-int} = & \min_j \overline{x}_{i,j}, \ i = 1, ..., \overline{m} \end{pmatrix}$.
Also $\mathbf{Y}_{I,t} = \begin{pmatrix} I_i^{t.ext-out} = & \max_j y_{i,t,j}, \ i = 1, ..., s \\ I_i^{t.int-out} = & \max_j h_{i,t,j}, \ i = 1, ..., s \end{pmatrix}$. IF frontier is defined to measure the efficiency of DMUs in each time period by comparing to the ideal frontier.
Let $G^{ext.out} = (G_i^{ext.out}, \ i = 1, ..., s)$ is defined as the global ideal vector of external outputs in which $G_i^{ext.out} = \max_{t \in W} I_i^{t.ext-out}, \ \forall i$. Also $G^{int.out} = (G_i^{int.out}, \ i = 1, ..., s)$ is defined as the global ideal vector of internal outputs in which $G_i^{int.out} = \max_{t \in W} I_i^{t.int-out}, \ \forall i$. In order to evaluate the efficiency of each DMU in the time period t, the following problem is developed:

$$
\begin{aligned}
D^{IF}(X_p^t, Y_p^t) = \max \quad & \sum_{i=1}^{s} (\alpha_{i,p}^t + \overline{\alpha}_{i,p}^t) && (6) \\
\text{s.t.} \quad & \sum_{t \in W} \lambda_t I_i^{t.var-int} \le x_{i,t,p}, && i = 1, ..., m, \\
& \sum_{t \in W} \lambda_t I_i^{t.cap-int} \le A_{t,p} k_{i,t-1,p}, && i = 1, ..., s, \\
& \sum_{t \in W} \lambda_t I_i^{t.qua-int} = \overline{x}_{i,p}, && i = 1, ..., \overline{m}, \\
& \sum_{t \in W} \lambda_t I_i^{t.ext-out} \ge y_{i,t,p} + \alpha_{i,p}^t R_{y_{i,t,p}}^{IF}, && i = 1, ..., s, \\
& \sum_{t \in W} \lambda_t I_i^{t.int-out} \ge h_{i,,t,p} + \overline{\alpha}_{i,p}^t R_{h_{i,t,p}}^{IF}, && i = 1, ..., s, \\
& \lambda_t \in \Lambda^t,
\end{aligned}
$$

in which Λ^t might be as one of the following set:

$$\Lambda^t_{VRS} = \{\lambda_t | \sum_{t\in W} \lambda_t = 1,\ \lambda_t \geq 0,\ \forall t \in W\}, \tag{7}$$

$$\Lambda^t_{NDRS} = \{\lambda_t | \sum_{t\in W} \lambda_t \geq 1,\ \lambda_t \geq 0,\ \forall t \in W\},$$

$$\Lambda^t_{NIRS} = \{\lambda_t | \sum_{t\in W} \lambda_t \leq 1,\ \lambda_t \geq 0,\ \forall t \in W\},$$

$$\Lambda^t_{CRS} = \{\lambda_t |\ \lambda_t \geq 0,\ \forall t \in W\}.$$

Also the $R^{IF}_{y_{i,t,p}} = G^{ext.out}_i - y_{i,t,p},\ \forall i$ and $R^{IF}_{h_{i,t,p}} = G^{int.out}_i - h_{i,t,p},\ \forall i$ are defined as improvement directions of external outputs and internal outputs respectively.

4 Concluding remarks

In this paper we introduced one method to rank DMUs which are assessed not only in one assessment period, but also they are assessed over a few non-separate assessment periods. To this end in each assessment period t, we defined ideal DMU that was a virtual unit. Then based on those ideal DMUs, we develop production possibility set whose efficiency frontier called ideal frontier, was mentioned for which all DMUs were compared to that. In order to do comparison each DMU with the ideal frontier, an improving vector was developed by which each DMU moved step by step along that vector until reaching the ideal frontier.

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