A Modified ADM for Solving Systems of Linear Fredholm Integral Equations of the Second Kind

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Abstract

In this paper, a modification of Adomian decomposition method (MADM) for solving a system of linear Fredholm integral equations of the second kind has been introduced. This new method is resulted from Adomian decomposition method (ADM) by a simple modification. To illustrate the MADM, one example is presented. Comparison of the result of applying the MADM and the ADM revealing the new technique is very effective and convenient.

Keywords: Adomian decomposition method; System of linear Fredholm integral equations of the second kind

1 Introduction

The Adomian decomposition method was first proposed by Adomian and used to solve a wide class of nonlinear and partial differential equations. The method is very powerful in finding the solutions for various physical problems. The advantage of the method is its fast convergence of the solution. This method has been applied to a wide class of deterministic and stochastic problems of mathematical and physical sciences. The method provides the solution in some rapidly convergent series with components that are elegantly computed [1, 2].

Since many physical problems are modeled by system of linear Fredholm integral equations of the second kind, the numerical solutions of such system of linear Fredholm integral equations of the second kind have been highly studied by many authors. In this paper, a simple modification on the ADM will be studied and will be applied to solve a system of linear Fredholm integral equations of the second kind. A system of linear Fredholm integral equations of the second kind can be considered as

\[ U(t) = F(t) + \int_{a}^{b} K(s, t)U(s)ds, \]
where

\[ U(t) = (u_1(t), \ldots, u_n(t))^t, \]
\[ F(t) = (f_1(t), \ldots, f_n(t))^t, \]
\[ K(s, t) = [k_{ij}(s, t)], \quad i = 1, \ldots, n, \quad j = 1, \ldots, n. \]

In Eq. (1), the functions \( K \) and \( F \) are given, and \( U \) is the vector function of the solution that will be determined [8]. Also, we suppose that the system (1) has a unique solution. However, the necessary and sufficient conditions for existence and uniqueness of the solution of the system (1) could be found in [8].

The rest of this paper is organized as follows: Section 2 is assigned to a brief introduction the ADM for the system of linear Fredholm integral equations of the second kind. In Section 3, proposed the modified technique. To illustrate and show the efficiency of the method one example is presented in Section 4. And conclusions will appear in Section 5.

## 2 ADM for the system (1)

In this section, we apply the ADM for the system of linear Fredholm integral equations of the second kind. To this end, consider the system of (1) as

\[ u_i(t) = f_i(t) + \int_a^b \sum_{j=1}^n k_{ij}(s, t)u_j(s)ds, \quad i = 1, \ldots, n. \]  

(2)

To solve (1) by the ADM [10], let

\[ u_i(t) = \sum_{m=0}^{\infty} u_{im}(t), \quad i = 1, \ldots, n. \]  

(3)

Substituting (3) into (2), we obtain

\[ \sum_{m=0}^{\infty} u_{im}(t) = f_i(t) + \int_a^b \sum_{j=1}^n k_{ij}(s, t)(\sum_{m=0}^{\infty} u_{jm}(s))ds, \quad i = 1, \ldots, n. \]  

(4)

Based on the recursion scheme of the ADM, we define

\[ u_{i,0}(t) = f_i(t), \quad i = 1, \ldots, n, \]  

(5)

and then

\[ u_{i,k+1}(t) = \int_a^b \sum_{j=1}^n k_{ij}(s, t)u_{jk}(s)ds, \quad i = 1, \ldots, n, \quad k = 0, 1, \ldots \]  

(6)
A modified ADM for solving systems

It is clear that better approximations can be obtained by evaluating more components of the decomposition series solution $u_i(t)$. We approximate $u_i(t)$ by

$$
\varphi_{ik}(t) = \sum_{m=0}^{k-1} u_{im}(t), \quad \text{where} \quad \lim_{k \to \infty} \varphi_{ik}(t) = u_i(t).
$$

We note here that the convergence question of this technique has been formally proved and justified by [9]. In the next section, we propose a simple modification such that increase the rate of the convergence.

3 Description of the modified technique

In this section, we propose a simple modification to accelerate convergence rate of the ADM applied to a systems of linear Fredholm integral equations of the second kind. In the ADM, the first terms of $u_i(t)$ is defined the known function $f_i(t)$. In modification technique, we define the first terms of (3) as follows

$$
\begin{align*}
\left\{ \begin{array}{l}
    u_{1,0}(t) = f_1(t), \\
    u_{i,0}(t) = f_i(t) + \int_a^b \sum_{j=1}^{i-1} k_{ij}(s,t) u_{j,0}(s) ds, \quad i = 2, \ldots, n.
\end{array} \right.
\end{align*}
$$

In other words, the values of $u_{i,0}(t)$, for $i = 2, \ldots, n$ are modified by previous values and the other terms of (3) are defined as follows

$$
\begin{align*}
\left\{ \begin{array}{l}
    u_{1,k+1} = \int_a^b \sum_{j=1}^n k_{i,j}(s,t) u_{j,k}(s) ds, \\
    u_{i,k+1} = \int_a^b (\sum_{j=1}^{i-1} k_{i,j}(s,t) u_{j,k+1}(s) + \sum_{j=i}^n k_{ij}(s,t) u_{j,k}(s)) ds, \quad i = 2, \ldots, n, \quad k = 0, 1, \ldots
\end{array} \right.
\end{align*}
$$

in order to calculate each $u_{i,j}$, for $i = 2, \ldots, n$ and $j = 1, 2, \ldots$ it’s previous values in the same iteration are used.

For further illustration we now use this method with $n = 2$ for solving a second kind Fredholm integral equation system. Consider the following second kind Fredholm integral equation system

$$
\begin{align*}
\left\{ \begin{array}{l}
    u_1(t) = f_1(t) + \int_0^1 (k_{11}(s,t) u_1(s) + k_{12}(s,t) u_2(s)) ds, \\
    u_2(t) = f_2(t) + \int_0^1 (k_{21}(s,t) u_1(s) + k_{22}(s,t) u_2(s)) ds,
\end{array} \right.
\end{align*}
$$

To solve (10) by the ADM, let
\[
\begin{align*}
\begin{cases}
  u_1(t) = \sum_{m=0}^{\infty} u_{1m}(t), \\
  u_2(t) = \sum_{m=0}^{\infty} u_{2m}(t),
\end{cases}
\end{align*}
\] (11)

Substituting (11) into (10), we obtain

\[
\begin{align*}
\begin{cases}
  \sum_{m=0}^{\infty} u_{1m}(t) = f_1(t) + \int_0^t (k_{11}(s, t) \sum_{m=0}^{\infty} u_{1m}(s) + k_{12}(s, t) \sum_{m=0}^{\infty} u_{2m}(s)) ds, \\
  \sum_{m=0}^{\infty} u_{2m}(t) = f_2(t) + \int_0^t (k_{21}(s, t) \sum_{m=0}^{\infty} u_{1m}(s) + k_{22}(s, t) \sum_{m=0}^{\infty} u_{2m}(s)) ds,
\end{cases}
\end{align*}
\] (12)

Based on the recursion scheme of the MADM, we define

\[
\begin{align*}
\begin{cases}
  u_{10}(t) = f_1(t), \\
  u_{20}(t) = f_2(t) + \int_0^t k_{21}(s, t)u_{10}(s) ds,
\end{cases}
\end{align*}
\] (13)

and

\[
\begin{align*}
\begin{cases}
  u_{11}(t) = \int_0^t (k_{11}(s, t)u_{10}(s) + k_{12}(s, t)u_{20}(s)) ds, \\
  u_{21}(t) = \int_0^t (k_{21}(s, t)u_{11}(s) + k_{22}(s, t)u_{20}(s)) ds,
\end{cases}
\end{align*}
\] (14)

in order to, the values of \(u_{20}(t)\) and \(u_{21}(t)\) are modified by previous values and the other terms of (10) are defined as follows

\[
\begin{align*}
\begin{cases}
  u_{1,k+1} = \int_0^t (k_{11}(s, t)u_{1,k}(s) + k_{12}(s, t)u_{2,k}(s)) ds, \\
  u_{2,k+1} = \int_0^t (k_{21}(s, t)u_{1,k+1}(s) + k_{22}(s, t)u_{2,k}(s)) ds, \quad k = 1, 2, \ldots
\end{cases}
\end{align*}
\] (15)

that is, to calculate each \(u_{2,k+1}\), for \(k = 1, 2, \ldots\) it’s previous values in the same iteration are used.

\section{4 Numerical Example}

To give a clear overview of the modified method, we present the following example. We apply the ADM and the MADM for a system of linear Fredholm integral equations and compare the obtain results. The computations associated with example were performed using the mathematica 7.
Example. Consider the following system of linear Fredholm integral equations [10]

\[
\begin{align*}
  u_1(t) &= f_1(t) + \int_0^t \frac{s+1}{3}(u_1(s) + u_2(s))ds, \\
  u_2(t) &= f_2(t) + \int_0^t st(u_1(s) + u_2(s))ds,
\end{align*}
\]

where \( f_1(t) = \frac{t}{18} + \frac{17}{36} \) and \( f_2(t) = t^2 - \frac{19}{12}t + 1 \). The exact solutions are \( u_1(t) = t + 1 \) and \( u_2(t) = t^2 + 1 \).

i) Adomian decomposition method

Using the ADM, we would have the following procedure

\[
\begin{align*}
  u_{1,0}(t) &= \frac{t}{18} + \frac{17}{36}, \\
  u_{2,0}(t) &= t^2 - \frac{19}{12}t + 1,
\end{align*}
\]

and

\[
\begin{align*}
  u_{1,m+1}(t) &= \int_0^t \frac{s+t}{3}(u_{1m}(s) + u_{2m}(s))ds, \\
  u_{2,m+1}(t) &= \int_0^t st(u_{1m}(s) + u_{2m}(s))ds, \quad m = 0, 1, \ldots
\end{align*}
\]

Eleven terms approximations to the solutions are derived as [9]

\[
\begin{align*}
  \varphi_{1,11}(t) &= u_{1,0}(t) + u_{1,1}(t) + \ldots + u_{1,10}(t) = 0.9813t + 0.9885, \\
  \varphi_{2,11}(t) &= u_{2,0}(t) + u_{2,1}(t) + \ldots + u_{2,10}(t) = t^2 - 0.03450t + 1.
\end{align*}
\]

ii) Modify Adomian decomposition method

Using the modified recursive relation (7) and (8), we obtain

\[
\begin{align*}
  u_{1,0}(t) &= \frac{t}{18} + \frac{17}{36}, \\
  u_{2,0}(t) &= t^2 - \frac{19}{12}t + 1 + \int_0^t stu_{10}(s)ds,
\end{align*}
\]

and
\[
\begin{align*}
  &u_{1,m+1}(t) = \int_0^1 s^2 t (u_{1m}(s) + u_{2m}(s)) ds, \\
  &u_{2,m+1}(t) = \int_0^1 s t (u_{1,m+1}(s) + u_{2m}(s)) ds, \quad m = 0, 1, \ldots
\end{align*}
\]

The approximated solution with eleven terms are
\[
\phi_{1,11}(t) = u_{1,0}(t) + u_{1,1}(t) + \ldots + u_{1,10}(t) = 0.993773 t + 0.99619,
\]
\[
\phi_{2,11}(t) = u_{2,0}(t) + u_{2,1}(t) + \ldots + u_{2,10}(t) = t^2 - 0.0088248t + 1.
\]

Absolute errors of the ADM and the MADM for some values of \( t \) are presented in Table 1.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( e_{ADM}(\phi_{1,11}(t)) )</th>
<th>( e_{MADM}(\phi_{1,11}(t)) )</th>
<th>( e_{ADM}(\phi_{2,11}(t)) )</th>
<th>( e_{MADM}(\phi_{2,11}(t)) )</th>
</tr>
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<tr>
<td>0</td>
<td>( 1.150E-2 )</td>
<td>( 3.803E-3 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>( 1.336E-2 )</td>
<td>( 4.425E-3 )</td>
<td>( 3.450E-3 )</td>
<td>( 8.828E-4 )</td>
</tr>
<tr>
<td>0.2</td>
<td>( 1.523E-2 )</td>
<td>( 5.047E-3 )</td>
<td>( 6.901E-3 )</td>
<td>( 1.765E-3 )</td>
</tr>
<tr>
<td>0.3</td>
<td>( 1.710E-2 )</td>
<td>( 5.670E-3 )</td>
<td>( 1.035E-2 )</td>
<td>( 2.648E-3 )</td>
</tr>
<tr>
<td>0.4</td>
<td>( 1.896E-2 )</td>
<td>( 6.293E-3 )</td>
<td>( 1.380E-2 )</td>
<td>( 3.531E-3 )</td>
</tr>
<tr>
<td>0.5</td>
<td>( 2.083E-2 )</td>
<td>( 6.915E-3 )</td>
<td>( 1.725E-2 )</td>
<td>( 4.414E-3 )</td>
</tr>
<tr>
<td>0.6</td>
<td>( 2.269E-2 )</td>
<td>( 7.538E-3 )</td>
<td>( 2.070E-2 )</td>
<td>( 5.297E-3 )</td>
</tr>
<tr>
<td>0.7</td>
<td>( 2.456E-2 )</td>
<td>( 8.161E-3 )</td>
<td>( 2.415E-2 )</td>
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<td>0.8</td>
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<td>( 2.760E-2 )</td>
<td>( 7.062E-3 )</td>
</tr>
<tr>
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<td>( 9.406E-3 )</td>
<td>( 3.105E-2 )</td>
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<td>( 3.016E-2 )</td>
<td>( 1.002E-2 )</td>
<td>( 3.450E-2 )</td>
<td>( 8.829E-3 )</td>
</tr>
</tbody>
</table>

5 Conclusion

In this paper, a modified form of ADM, for solving the systems of linear Fredholm integral equations of the second kind, is studied successfully. The modified method is better than ADM in the sense of accuracy and applicability. Illustrative example presented clearly support this claim. The results have been approved the efficiency of this method for solving these problems.
References


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