On the Spectra of Almost Periodic Symmetric Positive Definite Matrices Functions

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Abstract

In this note we prove that the eigenvalues of the almost periodic matrix function which is symmetric and positive definite are also almost periodic. An example is given to prove that the hypothesis of symmetry and positivity are not necessary.

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1 Introduction and preliminaries

Let us consider the system of linear differential equations

$$\frac{dx}{dt} = A(t)x$$

(1)

where $x \in \mathbb{C}$, $A(t)$ is an $n$-dimensional matrix, $t \in \mathbb{R}$, and we also suppose that $(t \mapsto A(t))$ is continuous function and it is Bohr almost periodic. Let $R_A(t)$ be the fundamental matrix for the system (1), $R_A(0) = I_n$, where $I_n$ is the identity matrix. However, the function $(t \mapsto R_A(t))$ need not to be almost periodic in $t$. Note that if the matrix is periodic in $t$ the statement is trivial taking into account Floquet’s theory ([1],[6]).

First, let us recall some notions on almost periodic functions. Let $E$ be a Banach space of dimension $p \in \mathbb{N}^*$ equipped with a norm $|.|$. Let $f$ be a complex valued continuous function defined on $\mathbb{R}$. A number $\tau$ is called an $\epsilon > 0$-almost period of $f$ if

$$|f(t + \tau) - f(t)|_\infty < \epsilon.$$
If for any $\epsilon > 0$ there exists a number $l(\epsilon)$ such that every interval of length $l(\epsilon)$ contains an $\epsilon$-almost period of $f$, then $f(.)$ is said to be almost periodic.

It is well known that the following three statements are equivalent:

- $f$ is an almost periodic function.
- The set of translations $f_h$ for $h \in \mathbb{R}$ forms a relatively compact set with respect to the uniform topology.
- $f$ is the uniform limit of a sequence of (generalized) trigonometric polynomials

$$P_m(x) = \sum_{n} \alpha_n e^{i\lambda_n x}, \lambda_n \in \mathbb{R}$$

Consequently, a Bohr almost periodic function is a continuous function which possesses very much almost periods. It is clear that continuous periodic functions are almost periodic. Besides, the function

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad t \mapsto \sin t + \sin \sqrt{2}t$$

is almost periodic. For more details on almost periodicity one can see for example ([3],[4],[5],[7]). This paper is devoted to the study of the cone $\Omega_{ap}$: the collection of the almost periodic functions $t \mapsto A(t)$ from $\mathbb{R}$ into $\mathcal{M}(n, \mathbb{R})$ such that for all $t \in \mathbb{R}$, $A(t)$ is symmetric positive definite. In particular, using the algebraic properties of exterior power, we shall demonstrate that for all $A(\cdot) \in \Omega_{ap}$ the eigenvalues of $A(\cdot)$ are almost periodic. This may not be surprising and, moreover, seems to be a widely accepted fact. Netherless, to our knowledge there exists no proof of this fact in the literature.

2 The cone $\Omega_{ap}$ of almost periodic matrix functions

We consider respectively on $\mathbb{R}^n$ and $\mathcal{M}(n, \mathbb{R})$ the norms

$$\|x\| = (\langle x, x \rangle)^{\frac{1}{2}}, \|A\| = \sup \{\|Ax\|, x \in \mathbb{R}^n, \|x\| = 1\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product of $\mathbb{R}^n$. For each integer $k, 0 \leq k \leq n$, the $k$-fold exterior power $\Lambda^k \mathbb{R}^n$ of $\mathbb{R}^n$ is the set of all formal expressions

$$\sum_{i=1}^{m} \alpha_i u_{i1} \wedge \cdots \wedge u_{ik}$$
where $\alpha_i \in \mathbb{R}, m \in \mathbb{N}$ and $u_{ij} \in \mathbb{R}^n$. For each matrix $A \in \mathcal{GL}(n, \mathbb{R})$ we define the automorphism $\Lambda^k A$ of $\Lambda^k \mathbb{R}^n$ by

$$\Lambda^k A : \Lambda^k \mathbb{R}^n \rightarrow \Lambda^k \mathbb{R}^n$$

$$u_1 \wedge \cdots \wedge u_k \mapsto Au_1 \wedge \cdots \wedge Au_k$$

Here $\Lambda^k A$ denotes the linear application above or its matrix in the canonical basis.

**Proposition 2.1.** [2]

1. $\forall c \in \mathbb{R}, u_1 \wedge \cdots \Lambda(cu_i) \wedge \cdots \Lambda u_k = cu_1 \wedge \cdots \Lambda u_i \wedge \cdots \Lambda u_k$

2. If $(e_1, \cdots, e_n)$ is a basis of $\mathbb{R}^n$, then $(e_{i_1}, \cdots, e_{i_k})_{1 \leq i_1 \leq \cdots \leq i_k \leq n}$ is a basis of $\Lambda^k \mathbb{R}^n$.

**Remark 2.2.** If we denote $\langle \cdot, \cdot \rangle$ the standard scalar product on $\mathbb{R}^n$ then we obtain a scalar product on $\Lambda^k \mathbb{R}^n$ by

$$\langle u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k \rangle = \det \left[ \langle u_i, v_j \rangle \right]_{i,j}.$$

Further, one can see that the real

$$\|u_1 \wedge \cdots \wedge u_k\|^2$$

is the Gram determinant. We denote by

$$\mathcal{GL}(n, \mathbb{R}) = \{ M \in \mathcal{M}(n, \mathbb{R}), \det(M) \neq 0 \}$$

and

$$\mathcal{O}(n, \mathbb{R}) = \{ M \in \mathcal{M}(n, \mathbb{R}), {}^t M M = I_n \}$$

### 3 Main result

In order to establish our main result several technical lemmas are introduced below.

**Lemma 3.1.** We have the following

1. If $A \in \mathcal{GL}(n, \mathbb{R})$ then $\Lambda^k A \in \mathcal{GL}(n, \mathbb{R})$.

2. If $A \in \mathcal{O}(n, \mathbb{R})$ then $\Lambda^k A \in \mathcal{O}(n, \mathbb{R})$.

3. If $A$ is symmetric then $\Lambda^k A$ is symmetric too.
Proof. First, one has
\[ A^{-1}A = AA^{-1} = I_{\mathbb{R}^n} \]
then
\[ \Lambda^k A^{-1}A = \Lambda^k A^{-1} \Lambda^k A = I_{\Lambda^k \mathbb{R}^n} \]
which proves that
\[ (\Lambda^k A)^{-1} = \Lambda^k A^{-1}. \]
For the second, let's \( A \in O(n, \mathbb{R}) \) then
\[ < \Lambda^k A (u_1 \wedge \cdots \wedge u_k), \Lambda^k A (v_1 \wedge \cdots \wedge v_k) > = < Au_1 \wedge \cdots \wedge Au_k, Av_1 \wedge \cdots \wedge Av_k > \]
\[ = \det \left[ \{ < Au_i, Av_j > \}_{i,j} \right] \]
\[ = \det \left[ \{ < u_i, v_j > \}_{i,j} \right] \]
\[ = < u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k >, \]
which proves that \( \Lambda^k A \in O(n, \mathbb{R}) \). Similarly, if the matrix \( A \) is symmetric then
\[ < \Lambda^k A (u_1 \wedge \cdots \wedge u_k), v_1 \wedge \cdots \wedge v_k > = < Au_1 \wedge \cdots \wedge Au_k, v_1 \wedge v_2 \wedge \cdots \wedge v_k > \]
\[ = \det \left[ \{ < Au_i, v_j > \}_{i,j} \right] \]
\[ = \det \left[ \{ < u_i, Av_j > \}_{i,j} \right] \]
\[ = < u_1 \wedge \cdots \wedge u_k, \Lambda^k A (v_1 \wedge \cdots \wedge v_k) >, \]
and hence \( \Lambda^k A \) is symmetric.

**Lemma 3.2.** Let's \( A \in GL(n, \mathbb{R}) \). Denote by \( \mu_1, \mu_2, \cdots, \mu_n \) the square roots of the eigenvalues of \( ^tAA \) then for all \( k \) \((1 \leq k \leq n)\) we have
\[ \| \Lambda^k A \| = \prod_{i=1}^{k} \mu_i. \]

Proof. Since \( A \in GL(n, \mathbb{R}) \) then \( A \) admits a polar decomposition ([9],[8]), there exists an orthogonal matrix \( O \) and a symmetric positive definite matrix \( S \) such that \( A = OS \). Note that \( S = P^{-1}DP \) where \( P \) is an orthogonal invertible matrix and \( D = \text{diag} (\mu_1, \mu_2, \cdots, \mu_n) \), with
\[ (\mu_1^2, \mu_2^2, \cdots, \mu_n^2) = Sp(^tAA) \]
And then $A = OP^{-1}DP = UDP$, where $U = OP^{-1} \in O(n, \mathbb{R})$. By lemma 1, $\Lambda^k U$ and $\Lambda^k P$ are in $O(n, \mathbb{R})$. Consequently

$$
\|\Lambda^k A\| = \|\Lambda^k U \Lambda^k D \Lambda^k P\|
$$

$$
= \sup \left\{ \|\Lambda^k D z\|, z \in \Lambda^k \mathbb{R}^n, \|z\| = 1 \right\}
$$

$$
= \sup \left\{ \mu_{i_1} \mu_{i_2} \cdots \mu_{i_k}; 1 \leq i_1 < i_2 < \cdots < i_k \leq i_n \right\}
$$

$$
= \prod_{i=1}^{n} \mu_i.
$$

**Lemma 3.3.** If $t \mapsto A(t)$ is almost periodic then for all $k (1 \leq k \leq n)$ the matrix function $B_k : t \mapsto \Lambda^k A(t)$ is a almost periodic function.

Proof. We have $A(\cdot)$ is uniformly bounded; so

$$
\exists M > 0, \|A(t)\| \leq M.
$$

and since $t \mapsto A(t)$ is almost periodic

$$
\forall \epsilon > 0, \exists \epsilon > 0, \forall \alpha \in \mathbb{R}, \exists \delta \in [\alpha, \alpha + \epsilon], \|A(t + \delta) - A(t)\| \leq \frac{\epsilon}{2M},
$$

and then

$$
\|B_2(t + \delta) - B_2(t)\| = \|\Lambda^2 A(t + \delta) - \Lambda^2 A(t)\|
$$

$$
= \|A(t + \delta) \Lambda A(t + \delta) - A(t) \Lambda A(t)\|
$$

$$
\leq \|A(t + \delta)\| \|A(t + \delta) - A(t)\| \|A(t)\|
$$

$$
\leq \frac{\epsilon}{2} + \frac{\epsilon}{2}
$$

which proves that $t \mapsto B_2(t)$ is almost periodic. We prove in the same way that $t \mapsto B_2(t)$ is continuous and finally the result follows immediately by induction.

**Theorem 3.4.** Let’s $A(\cdot) \in \Omega_{ap}$, then the eigenvalues of $A(\cdot)$ are almost periodic.

Proof. Let’s

$$
\mu_1(t) \geq \mu_2(t) \geq \cdots \geq \mu_n(t) > 0,
$$

the square root of the eigenvalues of $^t A(t) A(t)$. Clearly

$$
\|A(t)\| = \mu_1(t).
$$
Since the norm is uniformed continuous then \( t \mapsto \|A(t)\| \) is almost periodic and then \( t \mapsto \mu_1(t) \) is almost periodic \([3]\).

By induction we have

\[
\|\Lambda^{k+1}A(t)\| = \prod_{i=1}^{k+1} \mu_i = \mu_{k+1} \prod_{i=1}^{k} \mu_i.
\]

We conclude by induction and by the uniformity of the norm.

**Remark.** It should be mentioned that the hypothesis of symmetry and the positivity are not necessary. In fact, let us consider the next example:

\[
A(t) = \begin{pmatrix} 0 & \sin \alpha t + \sin \beta t \\ \sin \alpha t + \sin \beta t & -4 \end{pmatrix}.
\]

where \( \alpha \) and \( \beta \in \mathbb{R} \) such that \( \frac{\alpha}{\beta} \notin \mathbb{Q} \). It is clear that \( t \mapsto A(t) \) is almost periodic and for all \( t \in \mathbb{R} \) the eigenvalues of \( A(t) \) are

\[
\mu_1(t) = -2 - \sqrt{4 - (\sin \alpha t + \sin \beta t)^2} \quad \text{and} \quad \mu_2(t) = -2 + \sqrt{4 - (\sin \alpha t + \sin \beta t)^2}.
\]

**References**


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