Existence of Common Coupled Fixed Point for a Class of Mappings in Partially Ordered Metric Spaces

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Abstract
In this paper we introduce the class of P-contraction mappings, analogous to the concept of C-contraction [2]. Also, we obtain a fixed point result for this class of contractions in complete metric spaces.

Mathematics Subject Classification: 47H10, 54H25

Keywords: Fixed point, Multivalued mapping, Complete metric space

1 Introduction
In recent years, extension of the Banach’s contraction principle [2] has been considered by many authors in different metric spaces. In [3], Bhaskar and Lakshmikantham presented coupled fixed point results for mixed monotone operators in partially ordered metric spaces and in 2009, Lakshmikantham and Ciric [6] proved coupled coincidence and coupled common fixed point theorems for nonlinear contractive mappings in this spaces.

2 Main results

Definition 2.1 ([6]) Let $(X, \preceq, d)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$ be two self mappings. $F$ has the mixed $g$-monotone property if $F$ is monotone $g$-non-decreasing in its first argument and is monotone $g$-non-increasing in its second argument, that is, if for all $x_1, x_2 \in X$; $gx_1 \preceq gx_2$ implies $F(x_1, y) \preceq F(x_2, y)$ for any $y \in X$ and for all $y_1, y_2 \in X$; $gy_1 \succeq gy_2$ implies $F(x, y_1) \preceq F(x, y_2)$ for any $x \in X$. 
Definition 2.2 ([1]) The mappings $F : X \times X \to X$ and $g : X \to X$ are called $w$-compatible if $g(F(x, y)) = F(gx, gy)$, whenever $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Definition 2.3 ([6], [1]) An element $(x, y) \in X \times X$ is called:

1. a coupled coincidence point of mappings $F : X \times X \to X$ and $g : X \to X$ if $g(x) = F(x, y)$ and $g(y) = F(y, x)$, and $(gx, gy)$ is called coupled point of coincidence, and,

2. a common coupled fixed point of mappings $F : X \times X \to X$ and $g : X \to X$ if $x = g(x) = F(x, y)$ and $y = g(y) = F(y, x)$.

Theorem 2.4 Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that $F$ has the mixed $g$-monotone property and satisfy

$$d(F(x, y), F(u, v)) \leq \frac{1}{2}(d(gx, gu) + d(F(x, y), gy) + d(F(x, y), gu) + d(F(u, v), gx) + d(F(u, v), gu) - \varphi(d(gx, gu), d(F(x, y), gx), d(F(x, y), gu), d(F(u, v), gx), d(F(u, v), gu))),$$

for every two pairs $(x, y), (u, v) \in X \times X$ such that $gx \preceq gu$ and $gy \succeq gv$, where $\varphi : [0, \infty)^5 \to [0, \infty)$ be a continuous function such that $\varphi(x, y, z, t, u) = 0$ if and only if $x = y = z = t = u = 0$. Also suppose $X$ has the following properties:

i. If a non-decreasing sequence $x_n \to x$; then $x_n \preceq x$ for all $n \geq 0$.

ii. If a non-increasing sequence $y_n \to y$; then $y_n \succeq y$ for all $n \geq 0$.

Let $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subset of $X$. If there exists $(x_0, y_0) \in X \times X$ such that $gx_0 \leq F(x_0, y_0)$ and $gy_0 \geq F(y_0, x_0)$, then $F$ and $g$ have a coupled coincidence point in $X$.

Proof 2.5 Let $x_0, y_0 \in X$ be such that $gx_0 \leq F(x_0, y_0)$ and $gy_0 \geq F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, we can define $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$, then $gx_0 \leq F(x_0, y_0) = gx_1$ and $gy_0 \geq F(y_0, x_0) = gy_1$. Since $F$ has the mixed $g$-monotone property, we have $F(x_0, y_0) \preceq F(x_1, y_1) \preceq F(x_1, y_1)$ and $F(y_0, x_0) \succeq F(y_1, x_0) \succeq F(y_1, x_1)$. In this way we construct the sequences $z_n$ and $t_n$ inductively as $z_n = gx_n = F(x_{n-1}, y_{n-1})$, and $t_n = gy_n = F(y_{n-1}, x_{n-1})$, for all $n \geq 0$.

We know that for all $n \geq 0$, $z_{n-1} \preceq z_n$, and $t_{n-1} \succeq t_n$. This can be done as in Theorem 3.1. of [4], so we omit the proof of this part.

Step I. We will prove that $\lim_{n \to \infty} d(z_n, z_{n+1}) = \lim_{n \to \infty} d(t_n, t_{n+1}) = 0.$
Using 1 (which is possible since \(gx_{n-1} \leq gx_n\) and \(gy_{n-1} \geq gy_n\), we obtain that

\[
d(z_n, z_{n+1}) = d(F(x_{n-1}, y_{n-1}), F(x_n, y_n))
\leq \frac{1}{2} (d(gx_{n-1}, gx_n) + d(F(x_{n-1}, y_{n-1}), gx_{n-1}) + d(F(x_{n-1}, y_{n-1}), gx_n)
+ d(F(y_n, x_n), gx_{n-1}) + d(F(y_n, x_n), gx_n)) \leq 1
\]

\[
d(z_n, z_{n+1}) = \frac{1}{2} (d(z_n, z_{n+1}) + d(z_{n-1}, z_n) + d(z_{n+1}, z_{n-1} + d(z_{n+1}, z_n))
- \varphi (d(z_{n+1}, z_n), d(z_n, z_{n-1}), d(z_{n+1}, z_{n-1}))(z_{n+1}, z_n))
\leq \frac{1}{2}(d(z_{n+1}, z_n) + d(z_n, z_{n-1}) + d(z_{n+1}, z_{n-1} + d(z_{n+1}, z_n))
\leq \frac{1}{2}(3d(z_{n+1}, z_n) + 2d(z_n, z_{n-1})),
\]

(2)

hence, \(d(z_{n+1}, z_n) \leq d(z_n, z_{n-1})\).

Again, since \(gy_n \leq g_{y_{n-1}}\) and \(gx_n \geq g_{x_{n-1}}\),

\[
d(t_{n+1}, t_n) = d(F(y_n, x_n), F(y_{n-1}, x_{n-1}))
\leq \frac{1}{2} (d(gy_n, gy_{n-1}) + d(F(y_n, x_n), gy_n) + d(F(y_{n-1}, x_{n-1}), gy_{n-1})
+ d(F(y_{n-1}, x_{n-1}), gy_n) + d(F(y_n, x_n), gy_{n-1})) \leq 1
\]

\[
d(t_{n+1}, t_n) = \frac{1}{2} (d(t_{n+1}, t_n) + d(t_{n+1}, t_{n-1}) + d(t_n, t_n) + d(t_n, t_{n-1})
- \varphi (d(t_{n+1}, t_n), d(t_{n+1}, t_{n-1}), d(t_n, t_n), d(t_n, t_{n-1}))
\leq \frac{1}{2}(3d(t_{n+1}, t_n) + 2d(t_n, t_{n-1})),
\]

(3)

hence, \(d(t_{n+1}, t_n) \leq d(t_n, t_{n-1})\).

It follows that the sequences \(d(z_{n+1}, z_n)\) and \(d(t_{n+1}, t_n)\) are monotone decreasing sequences of non-negative real numbers and consequently there exist \(r, s \geq 0\) such that \(\lim_{n \to \infty} d(z_{n+1}, z_n) = r\), and \(\lim_{n \to \infty} d(t_{n+1}, t_n) = s\).

From 2 we have

\[
d(z_{n+1}, z_n) \leq \frac{1}{3} (d(z_n, z_{n+1}) + d(z_n, z_n) + d(z_{n+1}, z_n) + d(z_{n+1}, z_{n+1}))
\leq \frac{1}{3}(3d(z_{n+1}, z_n) + 2d(z_n, z_{n+1})).
\]

(4)

If \(n \to \infty\) in 4, we have \(r = \lim_{n \to \infty} \frac{1}{3} (3r + d(z_{n-1}, z_{n+1})) \leq r\), hence \(\lim_{n \to \infty} d(z_{n-1}, z_{n+1}) = 2r\).

We have proved in (2)

\[
d(z_n, z_{n+1}) \leq \frac{1}{3} (d(z_n, z_{n+1}) + d(z_n, z_{n+1}) + d(z_{n+1}, z_{n+1}) + d(z_{n+1}, z_n))
- \varphi (d(z_n, z_{n+1}), d(z_n, z_{n+1}), d(z_{n+1}, z_n), d(z_{n+1}, z_n))
\leq \frac{1}{3}(3d(z_{n-1}, z_n) + 2d(z_n, z_{n+1})).
\]

(5)
Now, if \( n \to \infty \) and since \( \varphi \) is continuous, we can obtain
\[
r \leq r - \varphi(r, r, 0, 2r) \leq r.
\]

Consequently, \( \varphi(r, r, 0, 2r) = 0 \). Hence
\[
\lim_{n \to \infty} d(z_{n+1}, z_n) = r = 0. \tag{6}
\]

In a same way, we have
\[
\lim_{n \to \infty} d(t_{n+1}, t_n) = s = 0. \tag{7}
\]

Now, we show that \( \{z_n\} \) and \( \{t_n\} \) are Cauchy sequences in \( X \).

Let \( \{z_n\} \) is not a Cauchy sequence, then there exists \( \varepsilon > 0 \) for which we can find subsequences \( \{z_{m(k)}\} \) and \( \{z_{n(k)}\} \) of \( \{z_n\} \) such that \( n(k) > m(k) > k \) and \( d(z_{m(k)}, z_{n(k)}) \geq \varepsilon \), where \( n(k) \) is the smallest index with this property, i.e.,
\[
d(z_{m(k)}, z_{n(k)-1}) < \varepsilon. \tag{8}
\]

From triangle inequality
\[
\varepsilon \leq d(z_{m(k)}, z_{n(k)}) \leq d(z_{m(k)}, z_{n(k)-1}) + d(z_{n(k)-1}, z_{n(k)}) < \varepsilon + d(z_{n(k)-1}, z_{n(k)}). \tag{9}
\]

If \( k \to \infty \), Since \( \lim_{n \to \infty} d(z_n, z_{n+1}) = 0 \), from 9 we can conclude that
\[
\lim_{k \to \infty} d(z_{m(k)}, z_{n(k)}) = \varepsilon. \tag{10}
\]

Moreover, we have
\[
|d(z_{n(k)}, z_{m(k)+1}) - d(z_{n(k)}, z_{m(k)})| \leq d(z_{m(k)+1}, z_{m(k)}), \tag{11}
\]
and
\[
|d(z_{n(k)+1}, z_{m(k)}) - d(z_{n(k)}, z_{m(k)})| \leq d(z_{n(k)+1}, z_{n(k)}), \tag{12}
\]
and
\[
|d(z_{m(k)+1}, z_{n(k)+1}) - d(z_{m(k)+1}, z_{n(k)})| \leq d(z_{n(k)+1}, z_{n(k)}). \tag{13}
\]

Since \( \lim_{n \to \infty} d(z_n, z_{n+1}) = 0 \), and 11, 12 and 13 are hold, we get
\[
\lim_{k \to \infty} d(z_{m(k)+1}, z_{n(k)}) = \lim_{k \to \infty} d(z_{m(k)+1}, z_{n(k)+1}) = \lim_{k \to \infty} d(z_{n(k)+1}, z_{m(k)}) = \varepsilon. \tag{14}
\]
Again, as \( n(k) > m(k) \), we have \( gx_{m(k)} \preceq gx_{n(k)} \), and \( gy_{m(k)} \succeq gy_{n(k)} \). So, from 1, for all \( k \geq 0 \), we have

\[
d(z_{m(k)+1}, z_{n(k)+1}) = d(F(x_{m(k)}, y_{m(k)}), F(x_{n(k)}, y_{n(k)}))
\]

\[
\leq \frac{1}{4} (d(gx_{m(k)}, gx_{n(k)}) + d(F(x_{m(k)}, y_{m(k)}), gx_{m(k)}) + d(F(x_{m(k)}, y_{m(k)}), gx_{n(k)}) + d(F(x_{n(k)}, y_{n(k)}), gx_{m(k)})
\]

\[
\leq \frac{1}{4} (d(gx_{m(k)}, gx_{n(k)}) + d(F(x_{m(k)}, y_{m(k)}), gx_{m(k)}) + d(F(x_{m(k)}, y_{m(k)}), gx_{n(k)}) + d(F(x_{n(k)}, y_{n(k)}), gx_{m(k)})
\]

\[
- \varphi(d(gx_{m(k)}, gx_{n(k)}), d(F(x_{m(k)}, y_{m(k)}), gx_{m(k)}), d(F(x_{m(k)}, y_{m(k)}), gx_{n(k)})
\]

\[
, d(F(x_{n(k)}, y_{n(k)}), gx_{m(k)}) + d(F(x_{n(k)}, y_{n(k)}), gx_{n(k)}))
\]

\[
= \frac{1}{8} (d(z_{m(k)}, z_{n(k)}) + d(z_{m(k)}+1, z_{n(k)}) + d(z_{m(k)}+1, z_{n(k)})
\]

\[
+ d(z_{n(k)+1}, z_{m(k)}) + d(z_{n(k)+1}, z_{n(k)})
\]

\[
- \varphi(d(z_{m(k)}, z_{n(k)}), d(z_{m(k)}+1, z_{m(k)}), d(z_{m(k)}+1, z_{n(k)})
\]

\[
, d(z_{n(k)+1}, z_{m(k)}), d(z_{n(k)+1}, z_{n(k)}))
\]

(15)

If \( k \to \infty \), from 10 and 14 we have, \( \varepsilon \leq \frac{1}{3}(3\varepsilon) - \varphi(\varepsilon, 0, \varepsilon, 0) \), hence, we have \( \varepsilon = 0 \), which is a contradiction and it follows that \( \{z_n\} \) is a Cauchy sequence in \( X \). Analogously, it can be proved that \( \{t_n\} \) is a Cauchy sequence in \( X \).

Since \( (X, d) \) is complete and \( \{z_n\} \) is Cauchy, there exists \( z \in X \) such that \( \lim_{n \to \infty} z_n = \lim_{n \to \infty} gx_n = z \), and since \( g(X) \) is closed and \( \{z_n\} \subseteq g(X) \), we have \( z \in g(X) \) and hence there exists \( u \in X \) such that \( z = gu \). Similarly, there exist \( t, v \in X \) such that \( t = \lim_{n \to \infty} t_n = \lim_{n \to \infty} gy_n = gv \).

We prove that \( (u, v) \) is a coupled coincidence point of \( F \) and \( g \).

We know that \( gx_n \) and \( gy_n \) are non-decreasing and non-increasing in \( X \), respectively and \( gx_n \to z = gu \) and \( gy_n \to t = gv \). From conditions of our theorem, \( gx_n \preceq gu \) and \( gy_n \succeq gv \). So, using 1 we obtain that

\[
d(z_{n+1}, F(u, v)) = d(F(x_n, y_n), F(u, v))
\]

\[
\leq \frac{1}{4} (d(gx_n, gu) + d(F(x_n, y_n), gx_n) + d(F(x_n, y_n), gu)
\]

\[
+ d(F(u, v), gx_n) + d(F(u, v), gu)
\]

\[
- \varphi(d(gx_n, gu), d(F(x_n, y_n), gx_n), d(F(x_n, y_n), gu)
\]

\[
, d(F(u, v), gx_n), d(F(u, v), gu))
\]

\[
= \frac{1}{8} (d(z_n, z) + d(z_{n+1}, z_n) + d(F(u, v), z_n) + d(F(u, v), z))
\]

\[
- \varphi(d(z_n, z), d(z_{n+1}, z_n), d(F(u, v), z_n), d(F(u, v), z))
\]

(16)

If in (16) \( n \to \infty \),

\[
d(z, F(u, v)) \leq \frac{1}{4} (d(z, z) + d(z, z) + d(z, z) + d(F(u, v), z))
\]

\[
- \varphi(d(z, z), d(z, z), d(z, z), d(F(u, v), z))
\]

and hence \( \varphi(0, 0, 0, d(F(u, v), z), d(F(u, v), z)) \leq -\frac{2}{3} d(z, F(u, v)) \leq 0 \), and therefore, \( d(z, F(u, v)) = 0 \). So, \( F(u, v) = z = g(u) \) and in a similar way
we can obtain that \( F(v, u) = t = g(v) \). That is, \( g \) and \( F \) have a coupled coincidence point.

**Theorem 2.6** Adding the following conditions to the hypotheses of Theorem 2.4, we obtain the existence of the common coupled fixed point of \( F \) and \( g \).

(i) If any nondecreasing sequence \( z_n \) in \( X \) converges to \( z \), then we assume \( gz \preceq z \), and also, if any nonincreasing sequence \( t_n \) in \( X \) converges to \( t \), then we assume \( gt \succeq t \).

(ii) \( g \) and \( F \) be \( w \)-compatible continuous mappings.

**Proof 2.7** We know that the nondecreasing sequence \( gx_n = z_n \rightarrow z \) and by our assumptions \( gz_n \preceq gz \preceq z = gu \).

Also, the nonincreasing sequence \( gy_n = t_n \rightarrow t \) and by our assumptions \( gt_n \succeq gt \succeq t = gv \).

So, from (1) we have

\[
d(F(z_n, t_n), F(u, v)) \leq \frac{1}{5} (d(gz_n, gu) + d(F(z_n, t_n), gz_n) + d(F(z_n, t_n), gu) \\
+ d(F(u, v), gz_n) + d(F(u, v), gu)) \\
- \varphi(d(gz_n, gu), d(F(z_n, t_n), gz_n), d(F(z_n, t_n), gu) \\
+ d(F(u, v), gz_n), d(F(u, v), gu)).
\] (18)

Since \( F \) and \( g \) are \( w \)-compatible, and \( F(u, v) = gu = z \) and \( F(v, u) = gv = t \) we have that \( gz = g(gu) = g(F(u, v)) = F(gu, gv) = F(z, t) \).

Now, if in (18), \( n \rightarrow \infty \), we obtain

\[
d(gz, z) \leq \frac{1}{5} (d(gz, z) + d(gz, gz) + d(gz, z) + d(z, gz) + d(z, z)) \\
- \varphi(d(gz, z), d(gz, gz), d(gz, z), d(z, gz), d(z, z)).
\] (19)

Hence, \( \varphi(d(gz, z), d(gz, gz), d(gz, z), d(z, gz), d(z, z)) = 0 \) and so, \( d(gz, z) = 0 \). Therefore \( gz = z \) and from \( F(z, t) = gz \), we conclude that \( F(z, t) = gz = z \).

Analogously, we can prove that \( F(z, t) = gt = t \).

Note that if \( (X, \preceq) \) be a partially ordered set, then we endow \( X \times X \) with the following partial order relation:

\[
(x, y) \preceq (u, v) \iff x \preceq u, y \succeq v.
\]

for all \( (x, y), (u, v) \in X \times X \). ([7])

**Theorem 2.8** Let all the conditions of theorem 2.6 be fulfilled.

\( F \) and \( g \) have a unique common coupled fixed point provided that the common coupled fixed points of \( F \) and \( g \) are comparable.
Proof 2.9 Let \((x, y)\) and \((u, v)\) be two common coupled fixed points of \(F\) and \(g\), i.e., \(x = g(x) = F(x, y), y = g(y) = F(y, x),\) and \(u = g(u) = F(u, v), v = g(v) = F(v, u)\).

Suppose that \((x, y)\) and \((u, v)\) are comparable.
Since \((u, v)\) is comparable with \((x, y)\), we may assume that \((x, y) \preceq (u, v)\).
Now, applying 1 one obtains that
\[
d(x, u) = d(F(x, y), F(u, v)) \\
\leq \frac{1}{2}(d(gx, gu) + d(F(x, y), gx) + d(F(x, y), gu) \\
+ d(F(u, v), gx) + d(F(u, v), gu)) \\
- \varphi(d(gx, gu), d(F(x, y), gx), d(F(x, y), gu)) \\
, d(F(u, v), gx), d(F(u, v), gu)) \tag{20} \\
= \frac{1}{2}(d(gx, gu) + 0 + d(gx, gu) + d(gu, gx) + 0) \\
- \varphi(d(gx, gu), 0, d(gx, gu), d(gu, gx), 0) \\
= \frac{1}{2}(d(x, u) + 0 + d(x, u) + d(u, x) + 0) \\
- \varphi(d(x, u), 0, d(x, u), d(u, x), 0).
\]

Therefore, \(\varphi(d(x, u), 0, d(x, u), d(u, x), 0) \leq -\frac{2}{5}d(x, u) \leq 0\). Hence \(x = u\).
In a similar way, we have \(y = v\).

References


Received: August, 2011