Cartesian Product of Bi Polar Q- Fuzzy Lattice

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Abstract. In this paper we introduced the notion of interval-valued bipolar Q-fuzzy lattices, briefly IVBQFL, in terms of fuzzy d-ideals based on bipolar valued fuzzy set and several related properties are also established. Relations between a interval-valued bipolar Q-fuzzy lattices and bipolar fuzzy d-ideals are also discussed. The concept of Cartesian product fuzzy fully invariant and characteristic of interval-valued bipolar Q-fuzzy lattices are investigated. The homomorphic image of bipolar Q-fuzzy lattice is also given. Furthermore, we state family of interval-valued bipolar Q-fuzzy lattices.

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1. Introduction

In the traditional fuzzy sets, the membership degrees of elements range over the interval [0, 1]. Bipolar valued fuzzy sets and intuitionistic fuzzy sets where introduced in 1986 [12] only with the membership degrees ranged on the interval [0, 1]. It is difficult to express the difference of the irrelevant elements from the contrary elements in fuzzy set. If a set representation could express this kind of difference, it would be more informative than the traditional fuzzy set representation. Based on these observations, K.M. Lee [8], introduced an extention of fuzzy sets named bipolar-valued fuzzy sets. He give two kinds of representations of the notion of bipolar-valued fuzzy sets in. Jun and Song [7] applied the notion of bipolar-valued fuzzy sets to BCH-Algebras. They introduced the concept of bipolar fuzzy sub algebras / ideals of a BCH algebras and investigated several properties. Interval-valued fuzzy sets were first introduced by the Zadeh [15] as a generalization of fuzzy sets. This idea gives the simplest method to capture the imprecision of the membership grades for a fuzzy sets. Thus interval-valued fuzzy set provide a more adequate description of uncertainty then the traditional fuzzy sets. It is therefore important to use interval-valued fuzzy sets in applications. One of the main application is in fuzzy control and the most computationally intensive part of fuzzy control is defuzzification. With the above background Atanassov and Gargov [1] introduce the notion of interval-valued intuitionistic fuzzy sets, which is a common generalization of intuitionistic fuzzy sets and interval-valued fuzzy sets. Therefore fuzzy sets are a kind of usefull Mathematical structure to represent a collection of objects whose boundary is vague. There are several kinds of fuzzy set extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval-valued fuzzy sets, vague set etc. Bipolar valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0, 1] to [-1, 1]. Bipolar valued fuzzy set have membership degrees that represent the degree of satisfaction to the property corresponding to a fuzzy set and its counter property. In a bipolar-valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degrees on [0, 1] indicate that elements somewhat satisfy the property, and the membership degrees on [-1, 0] indicate that elements somewhat satisfy the implicit counter –property (see [8]).
Figure 1 shows a bipolar-valued fuzzy set redefined for the fuzzy set “young”. The negative membership degrees indicate the satisfaction extent of elements to an implicit counter-property. (e.g. old against the property young). This kind of bipolar-valued fuzzy set representation enables the elements with membership degree 0 in traditional fuzzy sets, to be expressed into the elements with membership degree 0 (irrelevant elements). The age elements 50 and 95, with membership degree 0 in the fuzzy sets of figure 1, have 0 and a negative membership degree in the bipolar-valued fuzzy set of figure 1, respectively. Now it is manifested that 50 is an irrelevant age to the property young and 95 is more apart from the property young than 50, i.e. 95 is a contrary age to the property young see[8].

Figure 1. A bipolar fuzzy set “young”

In this paper, we introduce the notion of interval-valued bipolar fuzzy lattice, Cartesian product, fuzzy fully invariant lattice, characteristic and a homomorphic image of bipolar fuzzy lattice and then we investigate several properties. We give a characterization of bipolar fuzzy d-ideals.

2. PRELIMINARIES

Definition 2.1 :Let X be a non-empty set. A Q-fuzzy set in X is function \( \mu : X \times Q \rightarrow [0, 1] \).
Definition 2.2: Let G be a any non-empty set. A bipolar valued fuzzy set A in G is an object having the form $A = \{(x, \mu^+_A(x), \mu^-_A(x)) / x \in G\}$ Where $\mu^+_A : G \rightarrow [0, 1]$ and $\mu^-_A : G \rightarrow [-1, 0]$. are mapping. The positive membership degree $\mu^+_A(x)$ denotes the satisfaction degree of an element x to the property corresponding to “A” and the negative membership degree $\mu^-_A(x)$ denotes the satisfaction degree of x to some implicit counter property of A.

Definition 2.3: A Bipolar Lattice fuzzy set A in G is called Bipolar Q-fuzzy Lattice of G if it satisfies

\begin{align*}
\text{(BFL1)} & \quad \mu^+(x + y, q) \geq \min \{ \mu^+(x, q), \mu^+(y, q)\} \\
\text{(BFL2)} & \quad \mu^-(x + y, q) \leq \max \{ \mu^-(x, q), \mu^-(y, q)\} \\
\text{(BFL3)} & \quad \mu^-(x, q) \geq \mu^-(x, q) \text{ and } \mu^+(x, q) \leq \mu^+(x, q) \\
\text{(BFL4)} & \quad \mu^+(x y, q) \geq \min \{ \mu^+(x, q), \mu^+(y, q)\} \\
\text{(BFL5)} & \quad \mu^-(x y, q) \leq \max \{ \mu^-(x, q), \mu^-(y, q)\}
\end{align*}

for all $x, y \in G$.

Definition 2.4: A Lattice ordered Group (LG) is a system $G = (G, +, \leq)$

(i) $(G, +)$ is a group

(ii) $(G, \cdot)$ is a lattice

Definition 2.5: For a Bipolar fuzzy set A in and G $(\beta, \alpha) \in [-1, 0] \times [0, 1]$. We define $A^+_\beta = \{x \in X / \mu^+_A(x) \geq \alpha \}$, $A^-_\beta = \{x \in X / \mu^-_A(x) \geq \alpha \}$ Which are called positive t-cut and Negative s-cut of A respectively.

Definition 2.6: Let A be a Bipolar Lattice over a group G with $[x, y] \in A$. Then A is called interval-valued Bipolar fuzzy lattice over G if

\begin{align*}
\text{(IVBFL1)} & \quad \mu^+(x + y, q) \geq \min \{ \mu^+(x, q), \mu^+(y, q)\} \\
\text{(IVBFL2)} & \quad \mu^-(x + y, q) \leq \max \{ \mu^-(x, q), \mu^-(y, q)\} \\
\text{(IVBFL3)} & \quad \mu^-(x, q) \geq \mu^-(x, q) \text{ and } \mu^+(x, q) \leq \mu^+(x, q) \\
\text{(IVBFL4)} & \quad \mu^+(x y, q) \geq \min \{ \mu^+(x, q), \mu^+(y, q)\} \\
\text{(IVBFL5)} & \quad \mu^-(x y, q) \leq \max \{ \mu^-(x, q), \mu^-(y, q)\}
\end{align*}

for all $x, y \in G$.

Definition 2.7: A Bipolar fuzzy set (BFS) A in G is called Bipolar fuzzy d-ideal of G if it satisfies

\begin{align*}
\text{(BFD1)} & \quad \mu^+(x) \geq \min \{ \mu^+(xy), \mu^+(y)\} \text{ (BFD 2)} \mu^-(x) \leq \max \{ \mu^-(xy), \mu^-(y)\} \\
\text{(BFD 3)} & \quad \mu^+(x, y) \leq \min \{ \mu^+(x), \mu^+(y)\} \text{ (BFD 4)} \mu^-(x, y) \geq \max \{ \mu^-(x), \mu^-(y)\}, \text{ for all } x, y \in G.
\end{align*}

Definition 2.8: Let $\lambda$ and $\mu$ be two fuzzy subset in X. The Cartesian product $\lambda^+ \times \mu^+ : X \times X \rightarrow [0, 1]$ is defined by

$\lambda^+ \times \mu^+ (x, y) = T \{ \lambda^+(x), \mu^+(y) \} \text{ and}$
\[ \lambda^- \times \mu^- (x, y) = S \{ \lambda^- (x), \mu^- (y) \} \] for all \( x, y \in G \).

**Definition 2.9:** An Interval-valued Bipolar fuzzy d-ideal \( A \) of \( L \) is Fuzzy fully invariant if 
\[ \mu_1^+ (x) = \mu^+ (x) \] and 
\[ \mu_2^- (x) = \mu^- (x) \]

**Definition 2.10:** Let \( A \) be a Bipolar Fuzzy set in \( X \). The strongest Bipolar fuzzy relation on \( X \) that is fuzzy relation on \( A \) is \( \mu_A \) given by 
\[ \mu_A^+ (x, y) = T \{ A_i^+ (x), A_i^+ (y) \} \] 
\[ \mu_A^- (x, y) = S \{ A_i^- (x), A_i^- (y) \} \] for all \( x, y \in G \). For a sake of simplicity we shall use the symbol \( A = \{ \mu^+, \mu^- \} \) for the bipolar valued fuzzy set \( A = \{ ((x, \mu^+ (x), \mu^- (x)) / x \in G \} \) and for bipolar fuzzy regular lattice (BFRL) of \( G \).

**Proposition 3.3:** Let \( A \) and \( B \) are interval valued bipolar fuzzy lattice of \( L \) then \( A \times B \) is an interval valued bipolar fuzzy lattice of \( L \times L \).

**Proof:** Since \( A \) and \( B \) are interval valued bipolar fuzzy lattices and for all \( x, y \in L \) and we have
\[ (IVBFL1) \ (A \times B)^+ (x + y) = \inf \{ \inf \{ \min \{ A_i^+ (x + y), B_i^+ (x + y) \} \} \] 
\[ i \in \Lambda \]
\[ \geq \ \inf \{ \min \{ \min \{ A_i^+ (x), A_i^+ (y) \}, \min \{ B_i^+ (x), B_i^+ (y) \} \} \] 
\[ i \in \Lambda \]
\[ \geq \ \inf \{ \min \{ \inf A_i^+ (x), \inf A_i^+ (y) \}, \min \{ \inf B_i^+ (x), \inf B_i^+ (y) \} \} \] 
\[ i \in \Lambda \]
\[ \geq \ \min \{ \inf A_i^+ (x), \inf A_i^+ (y), \inf B_i^+ (x), \inf B_i^+ (y) \} \] 
\[ i \in \Lambda \]
\[ \geq \ \min \{ \min \{ A_i^+ (x), B_i^+ (x) \}, \min \{ A_i^+ (y), B_i^+ (y) \} \} \] 
\[ i \in \Lambda \]
\[ \geq \ \min \{ \min \{ \cup A_i^+ (x), \cup B_i^+ (x) \}, \min \{ \cup A_i^+ (y), \cup B_i^+ (y) \} \} \] 
\[ i \in \Lambda \]
\[ \geq \ \min \{ \cup (A \times B)_i^+ (x), \cup (A \times B)_i^+ (y) \} \] 
\[ i \in \Lambda \]
\[ \geq \ \min \{ (A \times B)^+ (x), (A \times B)^+ (y) \} \] 
\[ (IVBFL2) (A \times B)^- (x + y) = \sup \{ \max \{ A_i^- (x + y), B_i^- (x + y) \} \] 
\[ i \in \Lambda \]
\[ \leq \ \sup \{ \max \{ A_i^- (x), A_i^- (y) \}, \max \{ B_i^- (x), B_i^- (y) \} \} \] 
\[ i \in \Lambda \]
\[ \leq \max \{ \sup \{ \max \{ A_i^- (x), A_i^- (y) \}, \max \{ B_i^- (x), B_i^- (y) \}\} \} \]

\[ \leq \max \{ \max \{ \sup A_i^- (x), \sup A_i^- (y) \}, \{ \sup B_i^- (x), \sup B_i^- (y) \}\} \]

\[ \leq \max \{ \sup A_i^- (x), \sup B_i^- (x) \}, \{ \sup A_i^- (y), \sup B_i^- (y) \}\} \]

\[ \leq \min \{ \bigcap \{ A_i^- (x), \bigcap B_i^- (x) \}, \{ \bigcap A_i^- (y), \bigcap B_i^- (y) \}\} \]

\[ \leq \max \{ \bigcap (A \times B)^i_i^- (x), \bigcap (A \times B)^i_i^- (y) \}, \leq \max \{ (A \times B)^i_i^- (x), (A \times B)^i_i^- (y) \} \]

(IVBFL3) \[(A \times B)^+_i(i-x) = \sup ((A \times B)^+_i(i-x)) \]

\[ \geq \sup (A \times B)^+_i (x) \]

\[ \geq \bigcup (A \times B)^+_i (x) \]

\[ \geq (A \times B)^+_i (x) \]

and \[(A \times B)^-_i (x) = \sup ((A \times B)^-_i (x)) \]

\[ \leq \sup (A \times B)^-_i (x) \]

\[ \leq \bigcap (A \times B)^-_i (x) \]

\[ \leq (A \times B)^-_i (x) \]

(IVBFL4) \[(A \times B)^+_i [x, y] = \inf((A \times B)^+_i [x, y]) \]

\[ \geq \inf(\min \{ A_i^+_i [x, y], B_i^+_i [x, y] \}) \]

\[ \geq \inf \min \{ \min \{ A_i^+_i (x), A_i^+_i (y) \}, \min \{ B_i^+_i (x), B_i^+_i (y) \} \} \]

\[ \geq \inf \min \{ \min \{ A_i^+_i (x), A_i^+_i (y) \}, \inf A_i^+_i (y), \inf B_i^+_i (y) \} \}

\[ \geq \min \{ \bigcup A_i^+_i (x), \bigcup B_i^+_i (x) \}, \min \{ \bigcup A_i^+_i (y), \bigcup B_i^+_i (y) \} \}

\[ \geq \min \{ \bigcap (A \times B)^+_i (x), \bigcap (A \times B)^+_i (y) \} \]

\[ \geq \min \{ (A \times B)^+_i (x), (A \times B)^+_i (y) \} \]

(IVBFL5) \[(A \times B)^-_i [x, y] = \sup((A \times B)^-_i [x, y]) \]
\[ \leq \sup_{i \in \Lambda} \max \{ A_i^- [x, y], B_i^- [x, y] \} \]
\[ \leq \sup_{i \in \Lambda} \{ \max \{ A_i^- (x), A_i^- (y) \}, \max \{ B_i^- (x), B_i^- (y) \} \} \]
\[ \leq \max \{ \sup_{i \in \Lambda} A_i^- (x), \sup_{i \in \Lambda} B_i^- (x), \sup_{i \in \Lambda} A_i^- (y), \sup_{i \in \Lambda} B_i^- (y) \} \]
\[ \leq \max \{ \max \{ \sup A_i^- (x), \sup B_i^- (x) \}, \min \{ \sup A_i^- (y), \sup B_i^- (y) \} \} \]
\[ \leq \max \{ \min \{ \sup A_i^- (x), \sup B_i^- (x) \}, \min \{ \sup A_i^- (y), \sup B_i^- (y) \} \} \]
\[ \leq \max \{ \{ A \times B \}^+ (x), \{ A \times B \}^+ (y) \} \]

Hence \( A \times B \) is an interval valued bipolar fuzzy lattice.

**Proposition 3.4:** Let \( A \) and \( B \) be two interval valued bipolar fuzzy d-ideal lattices then \( A \times B \) is an interval valued bipolar fuzzy d-ideal lattices then \( A \times B \) is an interval valued bipolar fuzzy d-ideal lattice.

**Proof:** Since \( A \) and \( B \) are two interval valued bipolar fuzzy d-ideal lattice over \( G \).

(BFDI1) \( (A \times B)^+ (x) = \min \{ A^+ (x), B^+ (x) \} \)
\[ \leq \min \{ \min \{ A^+ (xy), A^+ (y) \}, \min \{ B^+ (xy), B^+ (y) \} \} \]
\[ \leq \min \{ \min \{ A^+ (xy), A^+ (xy) \}, \min \{ A^+ (y), B^+ (y) \} \} \]
\[ \leq \min \{ \{ (A \times B)^+ (xy) \}, (A \times B)^+ (y) \} \].

(BFDI2) \( (A \times B)^- (x) = \max \{ A^- (x), B^- (x) \} \)
\[ \geq \max \{ \max \{ A^- (xy), A^- (y) \}, \max \{ B^- (xy), B^- (y) \} \} \]
\[ \geq \max \{ \max \{ A^- (xy), A^- (xy) \}, \max \{ A^- (y), B^- (y) \} \} \]
\[ \geq \max \{ \{ (A \times B)^- (xy) \}, (A \times B)^- (y) \} \].

(BFDI3) \( (A \times B)^+ [x, y]= \min \{ A^+ [x, y], B^+ [x, y] \} \)
\[ \leq \min \{ \min \{ A^- (x), A^- (y) \}, \min \{ B^+ (x), B^+ (y) \} \} \]
\[ \leq \min \{ \min \{ A^- (x), B^- (x) \}, \min \{ A^- (y), B^- (y) \} \} \]
\[ \leq \min \{ \{ (A \times B)^+ (x) \}, (A \times B)^+ (y) \} \].

(BFDI4) \( (A \times B)^- [x, y]= \max \{ A^- [x, y], B^- [x, y] \} \)
\[ \geq \max \{ \max \{ A^- (x), A^- (y) \}, \max \{ B^- (x), B^- (y) \} \} \]
\[ \geq \max \{ \max \{ A^- (x), A^- (xy) \}, \max \{ B^- (x), B^- (y) \} \} \]
\( \geq \max \{ \max \{ A^{-}(x), B^{-}(x), A^{-}(y), B^{-}(y) \} \}
\geq \max \{ \max \{ A^{-}(x), B^{-}(x) \}, \max \{ A^{-}(y), B^{-}(y) \} \}
\geq \max \{ (A \times B)^{-}(x), (A \times B)^{-}(y) \}.

**Proposition 3.5:** If \( \{ A_i / i \in I \} \) is a family of interval valued bipolar fuzzy fully invariant lattice of \( L \), then \( \bigcap_{i \in I} A_i = (\bigwedge_{i \in I} \lambda^+_{A_i}, \bigvee_{i \in I} \lambda^-_{A_i}) \) is an interval valued bipolar fuzzy fully invariant lattice of \( L \). Where
\[
\lambda^+_{A_i}(x) = \inf \{ \lambda^+_{A_i}(x) / i \in I, x \in L \}
\]
\[
\lambda^-_{A_i}(x) = \sup \{ \lambda^-_{A_i}(x) / i \in I, x \in L \}
\]

**Proof:** It can be easily seen that \( \bigcap_{i \in I} A_i = (\bigwedge_{i \in I} \lambda^+_{A_i}, \bigvee_{i \in I} \lambda^-_{A_i}) \) is an interval valued bipolar fuzzy fully invariant lattice of \( L \). Let \( x \in G \) and \( f \in \text{end} \,(L) \). Then
\[
(\bigwedge_{i \in I} \lambda^+_{A_i})f(x) = (\bigwedge_{i \in I} \lambda^+_{A_i})(f(x)) = \inf \{ \lambda^+_{A_i}(f(x))/i \in I \}
\]
\[
\leq \inf \{ \lambda^+_{A_i}(x)/i \in I \} \leq (\bigwedge_{i \in I} \lambda^+_{A_i})(x)
\]

Also,
\[
(\bigvee_{i \in I} \lambda^-_{A_i})f(x) = (\bigvee_{i \in I} \lambda^-_{A_i})(f(x)) = \sup \{ \lambda^-_{A_i}(f(x))/i \in I \}
\]
\[
\geq \sup \{ \lambda^-_{A_i}(x)/i \in I \} \geq (\bigvee_{i \in I} \lambda^-_{A_i})(x)
\]

**Proposition 3.6:** If \( A \) is fully invariant interval valued bipolar fuzzy d-ideal then it is characteristic.

**Proof:** We can easily see that \( A \) is fully invariant interval valued bipolar fuzzy d-ideal of \( L \). Let \( x \in L \) and \( f \in \text{Aut}(G) \). If \( x \in A \) then \( f(x) \in f(A) \subseteq A \). Thus we have
\[
(i) \quad \lambda^+(f(x)) \leq \min \{ \lambda^+(xy), \lambda^+(y) \} \leq \min \{ \lambda^+(f(xy)), \lambda^+(f(y)) \}
\]
\[
= \min \{ \lambda^+(x), \lambda^+(y) \} \leq \lambda^-(x).
\]
\[
(ii) \quad \lambda^-(f(x)) \geq \max \{ \lambda^-(xy), \lambda^-(y) \} \geq \max \{ \lambda^-(f(xy)), \lambda^-(f(y)) \}
\]
\[
\geq \max \{ \lambda^-(x), \lambda^-(y) \} \geq \lambda^-(x).
\]
Cartesian product of bi polar Q-fuzzy lattice

(iii) \( \mu^+ [x, y] \leq \min \{ \mu^+(x), \mu^+(y) \} \leq \min \{ \mu^+(f(x)), \mu^+(f(y)) \} \)
\( \leq \min \{ \mu^+(x), \mu^+(y) \} \leq \mu^+ [x, y] \).

(iv) \( \mu^- [x, y] \geq \max \{ \mu^-(x), \mu^-(y) \} \geq \max \{ \mu^-(f(x)), \mu^-(f(y)) \} \)
\( \geq \max \{ \mu^-(x), \mu^-(y) \} \geq \mu^- [x, y] \).

**Proposition 3.7**: An interval valued bipolar fuzzy d-ideal is characteristic if and only if each of its level set is a characteristic d-ideal.

**Proof**: Let an interval valued bipolar fuzzy d-ideal is characteristic. Let \( t_0 \in \text{Im}(A) \), \( f \in \text{Aut}(G) \), \( x \in U(A; t_0) \), then \( A(f(x)) = A(x) \geq t_0 \), which mean that \( f(x) \in U(A; t_0) \).
Thus \( f(U(A; t_0)) \subseteq U(A; t_0) \) Since for each \( x \in U(A; t_0) \) there exists \( y \in G \) such that \( f(y) = x \), we have \( A(y) = A(f(y)) = A(x) \geq t_0 \). Hence, we conclude that \( y \in U(A; t) \). Consequently, \( x = f(y) \in f(U(A; t)) \).
Further, \( f(U(A; t)) = U(A; t) \), This proves that \( U(A; t) \) are characteristic. Conversely, If all level sets of \( A \) are characteristic fuzzy d-ideals of \( G \), then for \( x \in L, f \in \text{Aut}(G) \) and \( A(x) = t_0 \), then we have \( x \in U(A; t) \). Thus, \( f(x) \in f(U(A; t)) = U(A; t) \) And \( f(x) \in U(A; t_1) \) \( = f(U(A; t_1)) \) Where \( x \in U(A; t_1) \). This is a contradiction. Thus \( A(f(x)) = A(x) \) So \( A \) is characteristic.

**Proposition 3.8**: Let \( f: X \to Y \) be an epimorphism of lattices. If \( \mu^f \) is bipolar fuzzy lattice of \( X \) then is bipolar fuzzy lattice of \( Y \).

**Proof**: Let \( y \in Y \) there exist \( x \in X \) such that \( f(x) = y \), then
\( \mu^+(y) = \mu^+(f(x)) = \mu^+(f(e)) = \mu^+(e) \)
and \( \mu^-(y) = \mu^-(f(x)) = \mu^-(f(e)) = \mu^-(e) \).

Let \( x, y \in Y \), then there exist \( a, b \in X \) such that \( f(a) = x \) and \( f(b) = y \).
It follows that

(BFL1) \( \mu^+(x + y) = \mu^+(f(a)) = \mu^+(a) \) and \( \mu^-(x + y) = \mu^-(f(a)) = \mu^-(a) \)
\( \geq \min \{ \mu^+(a), \mu^-(b) \} \geq \min \{ \mu^+(f(a)), \mu^-(f(b)) \} \geq \min \{ \mu^+(x), \mu^-(y) \} \)

(BFL2) \( \mu^- (x + y) \leq \max \{ \mu^-(a), \mu^-(b) \} \leq \max \{ \mu^-(f(a)), \mu^-(f(b)) \} \)
\( \leq \max \{ \mu^- (x), \mu^- (y) \} \)

(BFL3) \( \mu^+ (-x) = \mu^+(f(-a)) = \mu^+(f(-a)) = \mu^+(f(a)) = \mu^+(x) \)
and \( \mu^- (-x) = \mu^- (f(-a)) = \mu^- (-a) = \mu^- (f(a)) = \mu^- (x) \)

(BFL4) \( \mu^+ [x, y] = \mu^+(f(a)) = \mu^+(a) \geq \min \{ \mu^+(a), \mu^+(b) \} \)
\[ \geq \min \{ \mu^+ (f(a)), \mu^+ (f(b)) \} \geq \min \{ \mu^+ (x), \mu^+ (y) \} \]
\[ (BFL5) \mu^- [x, y] = \mu^- (f(a)) = \mu^- (f(b)) \leq \max \{ \mu^- (a), \mu^- (b) \} \]
\[ \leq \max \{ \mu^- (f(a)), \mu^- (f(b)) \} \leq \max \{ \mu^- (x), \mu^- (y) \} \]

**Proposition 3.9**: Let \( A = (X, \mu^+_A, \mu^-_A) \) be a interval valued bipolar fuzzy d-ideal of \( X \). If the inequality \( xy \leq z \) holds in \( X \) then.

(i) \( \mu^+_A (x) \geq \min \{ \mu^+_A (y), \mu^+_A (z) \} \) (ii) \( \mu^-_A (x) \leq \max \{ \mu^-_A (y), \mu^-_A (z) \} \)

**Proof**: Let \( x, y, z \in X \) be such that \( xy \leq z \) then \( xy \leq z = 0 \) and so

(i) \( \mu^+_A (x) \geq \min \{ \mu^+_A (xy), \mu^+_A (y) \} \geq \min \{ \mu^+_A (xy)z, \mu^+_A (z) \} \)

(ii) \( \mu^-_A (x) \leq \max \{ \mu^-_A (xy), \mu^-_A (y) \} \leq \max \{ \mu^-_A (xy)z, \mu^-_A (z) \} \)

**Conclusions**

In this present paper, we have presented some properties of interval-valued bipolar fuzzy lattices (IVBFL) over bipolar fuzzy d-ideals. It is clear that the most of the results can be simply extended to interval-valued bipolar fuzzy lattices / ideals. The obtained results probably can be applied in various fields such as artificial intelligence, signal processing, multi agent-systems, robotics, genetic algorithms, decision making, automata theory and medical diagnosis. It is our hope that this work would serve as a foundation of furthermore study of the theory of soft algebra especially soft ideals. In our opinion the established results can be similarly extended to groups, rings and its generalizations, soft computing and applied in engineering problems.

**References**


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