Regional Domination in Distributed Systems

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Abstract

This work concerns the regional analysis of linear distributed systems. It is an extension of the notion of domination to the regional case. We define the weak and exact regional domination for input operators in the case of class of controlled systems. We give characterization results and the main properties of such relations. These notions and the considered approach are extended by duality, to observed systems and output operators. Analogous results and properties are established. The case of actuators and sensors is examined. Various situations are considered, applications and illustrative examples are also given.
Keywords: Region, domination, control, observation, actuators, sensors, distributed systems.

1 Introduction and problem statement:

This work concerns the regional analysis [1,4,...] of the notion of domination introduced in [2,3]. This notion consists to study the possibility to make a comparison between input or output operators.

Indeed, in the literature on the notions of control and observation for finite or infinite dimension systems, the main problems and the obtained results are focused on the fact if a system is (or not) controllable or observable [4,...,15]. Various situations and systems are considered, but the problem of possible comparison of the input and output parameters themselves, is not practically examined.

The notion of domination has been introduced firstly in [2] for lymped (finite dimension) systems. Concerning distributed systems, the notions of weak and exact domination are introduced in [3]. Hence, for controlled systems and input operators, the main properties of these relations, and their characterization results are given. Then, by duality, the considered approach and results are extended to observed systems and output operators. The case of actuators and sensors, as well as other situations, are examined.

In this paper which concerns also the regional analysis of distributed parameter systems, we give an extension to the regional case of these notions. We consider without loss of generality, a class of linear distributed systems described by a state equation of type:

\[
\begin{align*}
\dot{z}(t) &= Az(t) + Bu(t) \\
z(0) &= z_0
\end{align*}
\] (1)

where \( A \) generates a strongly continuous semi-group \( \{S(t)\}_{t \geq 0} \); \( B \in \mathcal{L}(U, Z) \); \( U \) is the control space, a Hilbert space; \( Z = L^2(\Omega) \) is the state space, \( \Omega \) is an open and bounded subset of \( \mathbb{R}^n \) with a sufficiently regular boundary. The system (1) is augmented with the following regional output equation

\[
(E_\omega) \ y_\omega(t) = C p^*_\omega p_\omega z(t)
\] (2)

where \( C \in \mathcal{L}(Z, Y) \), \( Y \) is the observation space, a Hilbert space; \( \omega \) is a
non empty subregion of $\Omega$, $p_\omega$ is the restriction operator defined by:

$$p_\omega : L^2(\Omega) \longrightarrow L^2(\omega)$$

$$z \longrightarrow p_\omega z = z|_\omega$$  (3)

$p_\omega^*$ is the adjoint operator of $p_\omega$. The regional observation at time $t$ is given by:

$$y_\omega(t) = C p_\omega^* p_\omega S(t) z_0 + C p_\omega^* p_\omega H_t(B) u$$  (4)

with

$$H_t(B) u = \int_0^t S(t - s) B u(s) ds$$  (5)

In fact, one can consider an output equation as follows

$$y_\omega(t) = C_\omega z(t) = C_\omega S(t) z_0 + C_\omega H_t(B) u$$  (6)

but the choice of an equation such (2) is made only for mathematical considerations in order to maintain the duality between the approaches and results concerning the input and output operators respectively.

The regional aspect of these notions is motivated by the fact that an input (or output) operator may dominates regionally another one in $\omega$, but not on the whole domain $\Omega$.

Here also, we give characterization results and the main properties related to the considered problem. The case of actuators and sensors [1, 2, 3, 4, 6, 9, 11, ...] is also examined. Applications and illustrative examples are equally presented.

2 Controlled systems and input operators

This section is focused on controlled systems and regional domination for input operators. Let $B_1 \in \mathcal{L}(U_1, Z)$ and $B_2 \in \mathcal{L}(U_2, Z)$ be two control operators, where $U_1$ and $U_2$ are two Hilbert spaces. The corresponding systems ($S_1$) and ($S_2$) are respectively defined by

$$(S_1) \left\{ \begin{array}{l}
\dot{z}_1(t) = A z_1(t) + B_1 u(t) ; \ 0 < t < T \\
z_1(0) = z_0
\end{array} \right.$$  (7)
and

\[(S_2) \left\{ \begin{array}{l}
\dot{z}_2(t) = A z_2(t) + B_2 v(t); \ 0 < t < T \\
z_2(0) = z_0 \end{array} \right. \quad (8)\]

We introduce hereafter the notions of weak and exact regional domination for input operators. We give the main properties and characterization results, and then an application to a diffusion system.

### 2.1 Weak and exact regional domination: Definitions, properties and characterization results

Let us first give the following definition.

**Definition 2.1**
We say that $B_1$ dominates $B_2$ (or that $(S_1)$ dominates $(S_2)$) in $\omega$ on $[0, T]$

i) exactly, if for any $v \in L^2(0, T; U_2)$, there exists a control $u \in L^2(0, T; U_1)$ such that

$$p_\omega H_T(B_1) u + p_\omega H_T(B_2) v = 0 \text{ in } L^2(\omega)$$

ii) weakly, if for any $\epsilon > 0$ and any $v \in L^2(0, T; U_2)$, there exists a control $u \in L^2(0, T; U_1)$ such that

$$\| p_\omega H_T(B_1) u + p_\omega H_T(B_2) v \|_{L^2(\omega)} < \epsilon$$

In these situations, we note respectively $B_2 \lesssim B_1$ (or $(S_2) \lesssim (S_1)$) and $B_2 \preceq B_1$ (or $(S_2) \preceq (S_1)$).

Let us note that also in the regional case, the relation of domination is reflexive, transitive but not symmetric (neither antisymmetric). We have the following characterization result.

**Proposition 2.2** $B_1$ dominates $B_2$ in $\omega$ on $[0, T]$

i) exactly if and only if

$$\operatorname{Im}(p_\omega H_T(B_2)) \subset \operatorname{Im}(p_\omega H_T(B_1))$$

or equivalently, there exists $\gamma > 0$ such that for any $\theta \in L^2(\omega)$, we have
\[ \| B^*_2 S^*(T \cdot \cdot p^*_2 \theta) \|_{L^2(0,T;U_2)} \leq \gamma \| B^*_1 S^*(T \cdot \cdot p^*_1 \theta) \|_{L^2(0,T;U_1)} \]

ii) weakly if and only if

\[ \text{Im}(p^*_\omega H_T(B_2)) \subset \text{Im}(p^*_\omega H_T(B_1)) \]

This is equivalent to

\[ \text{Ker}[(H_T(B_1))^* p^*_\omega] \subset \text{Ker}[(H_T(B_2))^* p^*_\omega] \]

In the case of actuators \((D_i, g_i)_{1 \leq i \leq p}\) and \((\Omega_j, h_j)_{1 \leq j \leq q}\), the operators \(B_1\) and \(B_2\) are defined as follows

\[ B_1 u(t) = \sum_{i=1}^{p} g_i u_i(t) \text{ and } B_2 v(t) = \sum_{j=1}^{q} h_j v_j(t) \]

where

\[ u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_p(t) \end{pmatrix} \text{ and } v(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_q(t) \end{pmatrix} \]

If \(B_2 \lesssim B_1\), we say that \((D_i, g_i)_{1 \leq i \leq p}\) are regionally more strategic than \((\Omega_j, h_j)_{1 \leq j \leq q}\) in \(\omega\).

We have the following properties.

**Proposition 2.3**

1) The regional exact domination implies the weak one.

2) \(B_1\) may dominates \(B_2\) regionally in \(\omega\), exactly or weakly, even if \(p < q\) or if the system \((S_1)\) is not controllable (exactly or weakly) in \(\omega\).

3) The exact (respectively weak) domination in \(\omega\) implies the exact (respectively weak) domination in any subregion \(\gamma \subset \omega\). In the two cases, the converse is not true.

4) Regionally, one actuator may be more strategic than \(q\) others with \(q > 1\).
5) ω-strategic actuators are regionally more strategic in ω than any finite number of actuators.

Now, in the case where the dynamics $A$ is defined by

$$Az = \sum_{n=1}^{+\infty} \lambda_n \sum_{j=1}^{r_n} \langle z, \varphi_{nj} \rangle \varphi_{nj} \tag{9}$$

where $\{\varphi_{nj}, j = 1, \ldots, r_n; n \in \mathbb{N}^\star\}$ is a complete orthonormal system of eigenfunctions associated to the real eigenvalues $(\lambda_n)_{n \geq 1}$ such that $\lambda_1 > \lambda_2 > \lambda_3 > \ldots$, $r_n$ is the multiplicity of $\lambda_n$, the s.c.s.g. $(S(t))_{t \geq 0}$ generated by $A$ is given by

$$S(t)z = \sum_{n=1}^{+\infty} e^{\lambda_n t} \sum_{j=1}^{r_n} \langle z, \varphi_{nj} \rangle \varphi_{nj}. \tag{10}$$

We have the following results.

**Proposition 2.4**

The system $(S_1)$ dominates $(S_2)$ in $\omega$ on $]0, T[$

i) exactly if and only if, there exists $\gamma > 0$ such that for any $z \in L^2(\omega)$, we have

$$\| \left( \sum_{n=1}^{+\infty} e^{\lambda_n t} \sum_{j=1}^{r_n} \langle z, \varphi_{nj} \rangle_{L^2(\omega)} \langle h_i, \varphi_{nj} \rangle \right)_{1 \leq i \leq q} \|_{L^2(0,T; \mathbb{R}^q)} \leq \gamma \| \left( \sum_{n=1}^{+\infty} e^{\lambda_n t} \sum_{j=1}^{r_n} \langle z, \varphi_{nj} \rangle_{L^2(\omega)} \langle g_i, \varphi_{nj} \rangle \right)_{1 \leq i \leq p} \|_{L^2(0,T; \mathbb{R}^p)}$$

ii) weakly if and only if, for any $n \geq 1$, we have

$$\text{Ker}[M_n P_{n,\omega}] \subset \text{Ker}[G_n P_{n,\omega}]$$

where for $n \geq 1$,

$$M_n = (\langle g_i, \varphi_{nj} \rangle)_{1 \leq i \leq p; 1 \leq j \leq r_n}$$

and

$$G_n = (\langle h_i, \varphi_{nj} \rangle)_{1 \leq i \leq q; 1 \leq j \leq r_n}$$

are the controllability matrices corresponding to the actuators $(D_i, g_i)_{1 \leq i \leq p}$ and $(\Omega_j, h_j)_{1 \leq j \leq q}$ respectively, $P_{n,\omega}$ is the linear mapping defined on $L^2(\omega)$ by
\[
\begin{align*}
P_{n,\omega}(z) = \begin{pmatrix}
\langle z, \varphi_{n,1} \rangle_{L^2(\omega)} \\
\vdots \\
\langle z, \varphi_{n,r_n} \rangle_{L^2(\omega)}
\end{pmatrix} \in \mathbb{R}^{r_n}
\end{align*}
\]

As an application, we examine hereafter the case of a diffusion system with a Dirichlet boundary condition.

### 2.2 Applications

We consider the following one dimension diffusion system

\[
(S_d) \begin{cases}
\frac{\partial z}{\partial t}(x, t) = \frac{\partial^2 z}{\partial x^2}(x, t) + g(x)u(t) & \text{in } ]0, a[ \times ]0, T[ \\
z(0, t) = z(a, t) = 0 & \text{on } ]0, T[ \\
z(x, 0) = 0 & \text{on } ]0, a[
\end{cases}
\]

(11)

In this case, \( r_n = 1, \varphi_n(x) = \sqrt{2} a \sin \left( \frac{n\pi x}{a} \right) \) and \( \lambda_n = -\frac{n^2 \pi^2}{a^2} \) for \( n \geq 1 \).

Let us note that in the particular case where \( a = 1, (\varphi_n = \sqrt{2} \ sin(n\pi \cdot))_n \) is a complete orthogonal system (orthonormal basis) of \( L^2(]0, 1[) \) and also that \( (\tilde{\varphi}_n = sin(2n\pi \cdot))_n \) is a complete orthogonal system of \( L^2(]0, \frac{1}{2}[, \frac{1}{2}[) \).

First, let us recall that an actuator \((]0, a[, g)\) is strategic if and only if, for any \( n \geq 1 \), we have

\( \langle g, \varphi_n \rangle \neq 0 \)

Hereafter, we give an example illustrating the following important property:

1. An actuator may be more strategic than another one, in a region \( \omega \), but not in the whole domain \( \Omega \).

**Example 2.5** Let \( \Omega = ]0, 1[ \) and suppose that the system \((S_1)\)

\( \text{corresponds to } (S_d) \text{ excited by an actuator } (\Omega, g) \text{ with a spatial distribution } g \text{ defined by} \)

\( g(x) = 1 - 2x \)
and that \((S_2)\) represents \((S_d)\) excited by an actuator \((\Omega, h)\) with a spatial distribution \(h = \varphi_1 + \varphi_2\). For \(z \in L^2(\omega)\), we have

\[
([H_T(B_1)]^\ast p_n^\ast z)(t) = \sum_{n=1}^{+\infty} e^{\lambda_n(t-T)} \langle z, \varphi_n \rangle_{L^2(\omega)} \langle g, \varphi_n \rangle ; \quad t \in ]0, T]\]

then

\[
z \in Ker([H_T(B_1)]^\ast p_n^\ast) \iff \forall n \in \mathbb{N}^*, \quad \langle z, \varphi_n \rangle_{L^2(\omega)} = 0
\]

Since

\[
\langle g, \varphi_n \rangle = \frac{\sqrt{2}}{\pi n} (1 + (-1)^n)
\]

we have

\[
z \in Ker([H_T(B_1)]^\ast p_2^\ast) \iff \forall n \in \mathbb{N}^*, \quad \langle z, \varphi_{2n} \rangle_{L^2(\omega)} = 0
\]

Using the fact that \((\tilde{\varphi}_n = \sin(2n\pi \cdot) = \frac{\sqrt{2}}{\pi} \varphi_{2n})_n\) is a complete orthogonal system of \(L^2([0, \frac{1}{2}])\), then \(z = 0\). Hence, the actuator \((]0, 1[, g)\) is \(\omega\)-strategic.

On the other hand, for every \(z \in L^2(\omega)\) and \(t \in ]0, T]\), we have

\[
([H_T(B_2)]^\ast p_n^\ast z)(t) = e^{\lambda_1(t-T)} \langle z, \varphi_1 \rangle_{L^2(\omega)} + e^{\lambda_2(t-T)} \langle z, \varphi_2 \rangle_{L^2(\omega)}
\]

then

\[
z \in Ker([H_T(B_2)]^\ast p_n^\ast) \iff \langle z, \varphi_1 \rangle_{L^2(\omega)} = \langle z, \varphi_2 \rangle_{L^2(\omega)} = 0
\]

Consequently

\(Ker([H_T(B_1)]^\ast p_n^\ast) \subset Ker([H_T(B_2)]^\ast p_n^\ast)\). But \(\varphi_1 \in Ker([H_T(B_1)]^\ast)\) and \(\varphi_1 \notin Ker([H_T(B_2)]^\ast)\), then the inclusion \(Ker([H_T(B_1)]^\ast) \subset Ker([H_T(B_2)]^\ast)\) is not true.
From the previous example, we deduce the following important result concerning the regional controllability.

**Corollary 2.6** A system may be regionally controllable, without being controllable on the whole domain $\Omega$.

We have also the following properties.

2. Regionally, an operator (actuator) may dominates another one weakly, but not exactly.

3. One actuator may dominates $p$ ($p > 1$) others.

4. There exist operators $B_1$ and $B_2$ such that any one of them do not dominates the other.

These properties and other situations are illustrated in [3] for $\omega = \Omega$.

3 Weak and exact regional domination for output operators

In this section related to observed systems and output operators, we consider the following autonomous system

$$ (S_0) \begin{cases} \dot{z}(t) = Az(t) \\ z(0) = z_0 \end{cases} $$

(12)

augmented with the following regional output equations:

$$ (E_{\omega}^j) \quad y_{\omega}^j(t) = C_j p_\omega^* p_\omega z(t) ; \quad j = 1, 2 $$

(13)

where for $j \in \{1, 2\}$, $C_j \in \mathcal{L}(Z, Y_j)$, $Y_j$ is an observation space, a Hilbert space. The observation $y_{\omega}^j(.)$ is given by

$$ y_{\omega}^j(t) = K_\omega(C_j)(t)z_0 $$

where

$$ K_\omega(C_j) = C_j p_\omega^* p_\omega S(.) $$
Its adjoint operator is given by
\[
[K_\omega(C_j)]^* y = \int_0^T S^*(t) p_\omega^* p_\omega C_j^* y(t) dt
\]

We have the following definitions.

**Definition 3.1** We say that the output operator \( C_1 \) dominates \( C_2 \) exactly (respectively weakly) in \( \omega \) on \([0, T] \), with respect to the system \((S_0)\), if

\[
\text{Im}([K_\omega(C_2)]^*) \subset \text{Im}([K_\omega(C_1)]^*)
\]

(respectively \( \text{Im}([K_\omega(C_2)]^*) \subset \text{Im}([K_\omega(C_1)]^*) \))

Let us note that for \( C_j = (B_j)^*; \quad j = 1, 2 \) and if we consider the dual systems

\[
(S_j) \begin{cases}
\dot{z}(t) = Az(t) + B_j u(t) ; \quad 0 < t < T \\
z(0) = z_0
\end{cases}
\]

and

\[
(S_j^*) \begin{cases}
\dot{z}(t) = A^* z(t) ; \quad 0 < t < T \\
y_j^*(t) = C_j^* p_\omega^* p_\omega z(t)
\end{cases}
\]

from the duality property, we deduce the following result.

**Proposition 3.2** The observed system \((S_1^*)\) dominates exactly (respectively weakly) \((S_2^*)\) in \( \omega \) on \([0, T] \), if and only if, the controlled system \((S_1)\) dominates exactly (respectively weakly) \((S_2)\) in \( \omega \) on \([0, T] \).

In the usual case where the observation is given by sensors, we define by the same sensors more strategic than other sensors in the region \( \omega \).

Using this duality property, we deduce from the previous section analogous characterization results and properties. On the other hand, one can deduce also that a system may be observable regionally, but not observable on the whole domain, and hence that sensors may be regionally strategic, but not strategic in \( \Omega \).
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