A Statistical Integral of Bohner Type

on Banach Space

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Abstract

In this paper we propose one type of Bohner integration in context of statistical convergence. This wants to construct a new convergence of functions in Banach space to definite the measurable functions. We proved the Egorov theorem to get the relations between st- measurability and strong st- measurability. The main result is construction one type of Bohner integral as the statistical integral. We give the example that there are functions that are integrable by this type integration and nonintegrable by classic definition and prove some convergence theorems.

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1. Introduction

It is known that the idea of statistical convergence was given by Zigmund [11] since 1935 in his well known work “Trigonometric series”. The concept of was formalized by Steinhaus[9] and Fast [2]. Some year latter the concept was reintroduced by Schoenberg [7]. Statistically convergence has become an active area of research in recent years. This concept appears in various studies about number theory, measure theory, summability theory, Banach space etc. In this presentation we follow concepts introduced by Fridy[4] about the convergence of sequences and the concept of Schoenberg about integration the basic concept is the statistically Cauchy convergence of Fridy[3]. On the Banach space we adopted this concept from the work of Conor etc.[1](1989). We generalized to the Banach space some results achieved by Gokhan [6] (2002) on real line and study a statistically type of Bochner integral.

2. Preliminaries

Let be An a subset of ordered natural set N. It said to have density δ(A) if
\[ \delta(A) = \lim_{n \to \infty} \frac{|A_n|}{n}, \]
where \( A_n = \{k<n : k \in A\} \) and with \(|A|\) denotes the cardinality of the set A. It is clear that the finite sets have the density zero and \( \delta(A^c) = 1-\delta(A) \) if \( A^c = N-A \). If a property \( P(k) = \{k : k \in A\} \) holds for all \( k \in A \) with \( \delta(A) = 1 \), we say that property P holds for almost all \( k \) that is a.a.k. The vektorial sequence \( x \) is statistically convergent to the vector(element) \( L \) of a vectorial normed space if for each \( \varepsilon > 0 \)
\[ \lim_{n \to \infty} \frac{1}{n} \{ k \in n : \|x_k - L\| \geq \varepsilon \} = 0 \]
i.e. \( \|x_k - L\| < \varepsilon \) a.a.k.
We write st-lim \( x_k = L \). In same manner, the sequence \( x \) is a statistically Cauchy sequence if for every \( \varepsilon > 0 \), there exists a number \( N = N(\varepsilon) \) such that
\[ \|x_k - x_N\| < \varepsilon \] a.a.k.
Now, we deals with generalization of pointwise statistically convergence of functions on normed space.
The sequence of functions \( \{f_k\} \) contains the functions with value in vectorial space. For each \( x \) of the domain we consider the functional sequence \( (f_k(x)) \). We
denote with S the set of x where the sequence \{f_k(x)\} converges. The function f defined as

\[ f(x) = \lim_{k \to \infty} f_k(x) ; x \in S \]

is called the limit function of the sequence \{f_k\}, we say that sequence \{f_k\} converges point wise to f for every x of S.

This means that for every point x in S and for every \( \varepsilon > 0 \), there exists N(\varepsilon) such that

\[ k > N \implies \|f_k(x) - f(x)\| < \varepsilon. \]

**Definition 1.** A sequence of functions \{f_k(x)\} is said to be pointwise statistically convergent to f if for every \( \varepsilon > 0 \)

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \{ k \leq n : \|f_k(x) - f(x)\| \geq \varepsilon, \forall x \in S \} = 0, \]

i.e. for every \( x \in S \), \( \|f_k(x) - f(x)\| < \varepsilon \).

a.a. k. We write \( \text{st-lim} \ f_k(x) = f(x) \) or \( f_k \rightarrow f \) on S.

This means that for every \( \delta > 0 \), there exists integer N such that

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \{ k \leq n : \|f_k(x) - f(x)\| \geq \varepsilon, \forall x \in S \} < \delta \]

For all \( n > N = (N(\varepsilon, \delta, x)) \) and for every \( \varepsilon > 0 \).

If the inequality in (1) holds for all but finitely many k, then usual limes, \( \lim_{k \to \infty} f_k(x) = f(x) \) on S. It follows that this limes implies \( \text{st-lim} f_k(x) = f(x) \). But the converse of this is not true.

**Example 2.**

We use the well known example that proves this one.

\[ x_k = \begin{cases} x & \text{k=m}^2 \\ 0 & \text{k \neq m}^2 \end{cases} \]

k=1,2,... is q divergent sequence in usual concept of limes of functions but it converges to zero by the statistically convergence.

Further, we denote \( (S, \Sigma, \mu) \) the probability measure space, where S is any set and \( \Sigma \) sigma algebra of Borel.

**Definition 3.** A function \( f : S \to X \), where X is a vectorial normed space is called simple function by \( \mu \), if there is a finite sequence measurable sets \{E_i\}, such that

\[ E_i \in S, i=1,...,n \quad E_i \cap E_j = \emptyset \quad \text{for i\neq j}, \quad S = \bigcup_{i=1}^{n} E_i \quad \text{and} \quad f(s) = x_i \quad \text{for} \ s \in E_i, \]
It represented in a form  $f = \sum_{i=1}^{n} x_i \chi_{E_i}$, where $\chi_{E_i}$ is a characteristic function of $E_i$.

We denote $T(\mu, X)$ –the set of simple functions with domain $S$. $T(\mu, X)$ is a vectorial space with the addition of simple function and multiple with the real number. The simple functions as it is known are the measurable functions.

**Definition 4.** The function $f: S \rightarrow X$ is called statistically measurable by $\mu$ on set $S$(in short form st-measurable) if there exists a sequence of simple functions $(f_n) \in T(\mu, X)$ that for every $s \in S$ and every $\varepsilon > 0$ holds:

$$\lim_{n \to \infty} \frac{1}{n} \left| \{ k \leq n : ||f_k(s) - f(s)|| \geq \varepsilon, \forall x \in S \} \right| = 0.$$ 

for almost all $s \in S$.

**Preposition 5.** A linear combination of st-measurable functions is st-measurable function.

**Proof.** Let $(f_n)$ and $(g_n)$ be two sequences of functions of the set $T(\mu, X)$ with domain $S$ such that st-lim $f_n(x) = f(x)$ and st-lim $g_n(x) = g(x)$ with $\alpha, \beta \in \mathbb{R}$. Let us observe the nontrivial case when $\alpha \neq 0$ and $\beta \neq 0$. Let $\varepsilon > 0$ be given and note that,

$$\{ k \leq n : ||\alpha f_n(x) + \beta g_n(x) - (\alpha f(x) + \beta g(x))|| \geq \varepsilon, \forall x \in S \} \subseteq \{ k \leq n : ||f_n(x) - f(x)|| \geq \frac{\varepsilon}{2|\alpha|}, \forall x \in S \} \cup \{ k \leq n : ||g_n(x) - g(x)|| \geq \frac{\varepsilon}{2|\beta|}, \forall x \in S \}.$$

Since

$$||\alpha f_n(x) + \beta g_n(x) - (\alpha f(x) + \beta g(x))|| \leq ||\alpha|| ||f_n(x) - f(x)|| + ||\beta|| ||g_n(x) - g(x)||,$$

Hence, we get st-lim $(\alpha f_n(x) + \beta g_n(x)) = \alpha f(x) + \beta g(x)$. □

**Definition 6.** The function $f: S \rightarrow X$ is called statistically strong measurable by $\mu$ on $S$ if every $\delta > 0$ and every $\varepsilon > 0$ there exists an integer $N(\varepsilon, \delta)$ such

$$\frac{1}{n} \left| \{ k \leq n : ||f_k(s) - f(s)|| \geq \varepsilon \} \right| < \delta$$

for $k > N(\varepsilon, \delta)$ almost for every $s \in S$. In this case it is said the sequence statistically strong converges almost everywhere uniformly by $\mu$ to the function $f$ on $S$.

Now we modify some techniques developed from [8] to prove the following preposition.

**Theorem 7.(Theorem Egorov).** If a function $f: S \rightarrow X$ is st-measurable by $\mu$, then it is st- strong measurable uniformly almost everywhere on $S$. 
**Proof:** Since the function \( f : S \to X \) is st-measurable by \( \mu \), then there exists the measurable set \( Z \in \Sigma \) with \( \mu(Z)=0 \) and the sequence of simple functions \( (f_n) \) such that for every \( s \in S/Z \) it converges statistically in pointwise way to \( f(s) \),

\[
\lim_{n \to \infty} \frac{1}{n} \{ k \leq n : \| f_k(s) - f(s) \| \geq \frac{1}{k} \} = 0
\]

We construct the sets

\[
E_{k,n} = \{ s \in S \setminus Z : \| f_n(s) - f(s) \| < \frac{1}{k}, \forall n \in A_n \} \quad \text{where} \quad A_n = N \setminus A_n
\]

It is clear that \( s \in E_{k,n} \supset E_{k,n+1} \)

Since the sequence \( (f_n(s)) \) converges to \( f(s) \) for every \( s \in S \setminus Z \) and every \( k \), we write

\[
\bigcup_{n=1}^{\infty} E_{k,n} \subset S \setminus Z \quad \text{or} \quad \bigcap_{n=1}^{\infty} (S \setminus E_{k,n}) \subset Z
\]

From this inequality follows that for every \( \varepsilon > 0 \) and every \( k \in N \) we can find an integer \( n_k \) such that

\[
\mu(S \setminus E_{k,n_k}) < \frac{\varepsilon}{2^k}.
\]

We set \( A_k = \bigcap_{k=1}^{\infty} E_{k,n_k} \) then

\[
\mu(S \setminus A_k) = \mu(S \setminus \bigcap_{k=1}^{\infty} E_{k,n_k}) = \mu(\bigcup_{k=1}^{\infty} (S \setminus E_{k,n_k})) \leq \varepsilon
\]

This yields that \( \| f_n(s) - f(s) \| < \frac{1}{k} \), for every \( k \in A_n \). This means that the sequence \( (f_n) \) statistically uniformly converges to \( f(s) \) on \( S \setminus A_k \).

\[\square\]

**3. The statistically integral of Bochner type**

**Definition 8.** The integral of the simple function \( f : S \to X \), is called the element of vectorial normed space \( \sum_{i=1}^{n} x_i \mu(E_i) \), symbolically

\[
\int_S f(s) d\mu = \sum_{i=1}^{n} x_i \mu(E_i)
\]

In case when \( E \) is a measurable set and \( E \subset S \), then integral of simple function \( f \) on \( E \) is the integral of function \( f \chi_E \), we write

\[
\int_E f(s) d\mu = \int_S (f \chi_E)(s) d\mu
\]
We define the map
\[ \| \cdot \|_T : T(\mu, E) \to P ; \| f \|_T = \int_S \| f(s) \| \, d\mu \, . \]

It is easy to prove that \( \| f \|_T \) is a seminorm.

Following the definition of Cauchy sequences introduced by Fridy [3] and their extension to the functional sequences (see for example to [6]). The sequence \((f_k)\) is called the statistically Cauchy sequence if for every \( \varepsilon > 0 \) there exits an integer \( N(\varepsilon, x) \) with
\[
\lim_{n \to \infty} \frac{1}{n} \{ k \leq n : \| f_k(x) - f_N(x) \| \geq \varepsilon \ \forall x \in S \} = 0
\]

On the set of st-Cauchy simple sequence we define the equivalence relation:
\[
(f_n) \sim (g_n) \iff \text{st} - \lim \| f_n-g_n \| = 0.
\]

We the following theorem we extend in case of Banach space the result presented in [6].

**Theorem 9.** [3], [6]

Let \((f_k)\) be a sequence of functions on a set \( S \) with value to Banach space \( X \). The following statements are equivalent:

a) the sequence \((f_k)\) is pointwise statistically convergent on \( S \);

b) the sequence \((f_k)\) is statistically Cauchy sequence on \( S \).

**Proof.** It easy to prove that a) implies b) because we can do as in classic case where every convergent sequence is a Cauchy sequence. Assume that \( \text{st} - \lim f_k(s) = f(s) \) on \( S \) and let be \( \varepsilon > 0 \). Then
\[
\lim_{n \to \infty} \frac{1}{n} \{ k \leq n : \| f_k(s) - f(s) \| \geq \varepsilon \} = 0
\]

or \( \| f_n(s)-f(s)\|<\frac{\varepsilon}{2} \) a.a.k and we choose an index \( N \) such that \( \| f_N(s)-f(s)\|<\frac{\varepsilon}{2} \).

The prove follows from the inequality
\[
\| f_k(s)-f(s)\| \leq \| f_k(s)-f(s)\| + \| f_N(s)-f(s)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \text{ a.a.k}
\]

and every \( s \in S \). Hence \((f_k)\) is statistically Cauchy.

Let suppose that b) holds and choose an index \( N \) such that closed ball \( B(f_N(s), 1) \) contains \( f_k(s) \) a.a.k. and every \( s \in S \). We use again b) and choose an index \( M \) such that the boule \( B'(f_M(s), \frac{1}{2}) \) contains \( f_k(s) \) a.a.k. and every \( s \in S \). Denote \( B_1 = B \cap B' \) it seems that \( B_1 \) contains \( f_k(s) \) a.a.k. and every \( s \in S \). We have
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\{ k \leq n : f_k(s) \notin B_1 \} \subset \{ k \leq n : f_k(s) \notin B \} \cup \{ k \leq n : f_k(s) \notin B' \}

so

$$\lim_{n \to \infty} \frac{1}{n} \left\{ k \leq n : f_k(s) \notin B_1 \text{ for every } s \in S \right\}$$

$$\leq \lim_{n \to \infty} \frac{1}{n} \left\{ k \leq n : f_k(s) \notin B \text{ for every } s \in S \right\}$$

$$+ \lim_{n \to \infty} \frac{1}{n} \left\{ k \leq n : f_k(s) \notin B' \text{ for every } s \in S \right\} = 0$$

We take here the close ball \( B_1 \) with radius less than or equal to 1 that contains \( f_k(s) \) a.a.k. and every \( s \in S \). Latter we proceed by choosing an index \( N(2) \) such that the closed \( B''(f_{N(2)}, \frac{1}{2^2}) \) contains \( f_k(s) \) a.a.k. and with the same argument the closed ball \( B_2 = B_1 \cap B'' \) contains \( f_k(s) \) a.a.k and every \( s \in S \) and radius of \( B_2 \) is less than or equal to \( \frac{1}{2} \). Continuing inductively we can construct the sequence of closed balls \( (B_m)_{m=1}^{\infty} \) such that for every \( m, B_m \supseteq B_{m+1} \) and radius of \( B_m \) is less than or equal to \( 2^{-m} \), and \( f_k(s) \in B_m \) a.a.k. and every \( s \in S \). Thus by virtue of completeness of Banach space \( X \), there exists a function \( f(s) \), defined on \( S \) such that \( \{ f \} \) is uniquely in \( \bigcap_{m=1}^{\infty} B_m \).

**Lemma 10.** If the sequence of simple functions \( (f_n) \) is a st-Cauchy sequence on Banach space than exists st-lim \( \int_S f_k(s)d\mu \).

**Proof.** We have that \( ||f_k(s) - f_N(s)|| < \frac{\varepsilon}{\mu(S)} \) a.a.k and an integer \( N \), so

\[
\| \int_S f_k(s)d\mu - \int_S f_N(s)d\mu \| \leq \int_S \| f_k(s) - f_N(s) \| d\mu \leq \| f_k(s) - f_N(s) \| \mu(S) < \varepsilon \text{ a.a.k and integer } N.
\]

We obtain that \( (\int_S f_n(s)d\mu) \) is a st-Cauchy sequence in \( X \) by the norm, it implies that the sequence is st-convergent.

Let \( X \) be a separable Banach space.

**Definition 11.** The function \( f : S \to X \) is called st-Bochner integrable if there exists a st-Cauchy sequence of simple functions \( (f_k) \) such that:

i) statistically convergent a.e. by \( \mu \) to the function \( f \);
ii) \( \text{st-lim}_{k} \int_{S} \| f_{k}(s) - f_{n}(s) \| d\mu = 0 \) a.e.

\( \text{st-lim}_{k} \int_{S} f_{n}(s)d\mu \) is called st-Bochner integral and denote with \((B_{s}) \int_{S} f(x)d\mu\)

This sequence \((f_{n})\) of simple functions is called determinant of function \(f\).

**Theorem 12.** If \((f_{n})\) and \((g_{n})\) as st-Cauchy sequences are determinants of the same function \(f\) than

\[
\text{st-lim}_{k} \int_{S} f_{n}(s)d\mu = \text{st-lim}_{k} \int_{S} g_{n}(s)d\mu
\]

**Proof.** The inequality \( \| f_{n}(s)-g_{n}(s) \| \leq \| f_{n}(s)-f(s) \| + \| f(s)-g_{n}(s) \| \)

shows that \((f_{n})\) and \((g_{n})\) are equivalent \(\text{st-lim}\|f_{n} - g_{n}\| = 0\) or for every \(\varepsilon > 0\) \(\|f_{k}(s) - g_{k}(s)\| < \varepsilon\) a.a.k.

By the definition of integral:

\[
\| \int_{S} f_{n}(s) - (\text{st-lim} \int_{S} f_{n}(s)d\mu) \| < \varepsilon \text{ and } \| \int_{S} g_{n}(s) - (\text{st-lim} \int_{S} g_{n}(s)d\mu) \| < \varepsilon
\]

a.a.k and every \(s \in S\).

Consider the difference

\[
\| (\text{st-lim} \int_{S} f_{n}(s)d\mu) - (\text{st-lim} \int_{S} g_{n}(s)d\mu) \| \leq
\| (\text{st-lim} \int_{S} f_{n}(s)d\mu) - \int_{S} f_{n}(s)d\mu \| + \| \int_{S} f_{n}(s)d\mu - \int_{S} g_{n}(s)d\mu \| + \| \int_{S} g_{n}(s)d\mu - (\text{st-lim} \int_{S} g_{n}(s)d\mu) \| < 3\varepsilon
\]

And we have proved the above equality.

It easy to watch that usual Bohner integral is a st-Bohner integral but the reveres is not true.

**Example 13.** Let \((f_{n})\) is a sequence defined by formula

\[
f_{k}(x) = \begin{cases} (k+1)(-x)^{k} & \text{for } k \in [3^{p}, 3^{p} + p], \ p = 1, 2, ... \\ 0 & \text{on the contrary} \end{cases}
\]

If we take \(x \in P \setminus [-1,1] \), \(k=1,2,...\) then

\[
\frac{1}{n} | \{k \leq n : f_{k}(x) \neq 0 \text{ whenever } x \in R \setminus [-1,1] \} | \leq \frac{p(p+1)}{3^{p+1}}
\]

So, we have that \(\text{st-lim} f_{k}(x) = 0\) or \(B_{s} - \int_{R \setminus [-1,1]} f_{k}(x)d\mu = 0\),

On the other hand, the usual integral is indefinite.
\[
\int_1^\infty (k + 1)(-x)^k \, dx = (-1)^k (B^{k+1} - 1) \to \pm \infty
\]

3. The property of statistically integral

**Theorem 14.** If the function \( f \) is st-Bohner integrable then the function \( \|f\| \) is also st-Bohner integrable.

**Proof.** Following definition of st-integrability, there exists the sequence of simple functions \( f_k \) convergent almost everywhere and a.a.k. to the function \( f \) and \( \int_S \|f_k(s) - f_N(s)\| \, d\mu < \varepsilon \) a.a.k. We consider the inequality \( \|f_k\| - \|f\| \leq \|f_k - f\| \).

Hence \( \text{st-lim } f_k(s) = f(s) \) a.e., it follows that \( \text{st-lim} \|f_k(s)\| = \|f(s)\| \) a.e.

Inequality
\[
\int_S \|f_k\| - \|f_N\| \, d\mu \leq \int_S \|f_k - f_N\| \, d\mu
\]

Shows that \( \|f\| \) is st-integrable.

The equality \( \text{st-lim} \int_A \int_S f_{n}d\mu = (Bs) \int_S f_{n}d\mu \), where \( (f_n) \) is sequence of simple functions determinant to \( f \), and the well known properties of classical integral allow us to formulate the following properties of st-Bohner integral:

(I) \( (Bs) \int_S (\alpha f(s) + \beta g(s)) \, d\mu = (Bs)\alpha \int_S f(s) \, d\mu + (Bs)\beta \int_S g(s) \, d\mu \)

(II) If \( A = A_1 \cup A_2, A_1 \cap A_2 = \emptyset \) and \( (f_n(s)) \) is a sequence of simple functions determinant of the function \( f(s) \) then
\[
\int_A f_n(s) \, d\mu = \int_{A_1} f_n(s) \, d\mu + \int_{A_2} f_n(s) \, d\mu.
\]

If we take the limes of above equality whenever \( n \to \infty \), we have
\[
(Bs) \int_A f(s) \, d\mu = (Bs) \int_{A_1} f(s) \, d\mu + (Bs) \int_{A_2} f(s) \, d\mu
\]

(III) \( (Bs) \int_S f \, d\mu \leq (Bs) \int_S f \, d\mu \)

This inequality we obtain from the same inequality for simple functions and isotonic property [9].
\[
\int_S f_n \, d\mu \leq \int_S f_n \, d\mu
\]
(IV) Applying the property (III) for the functions statistically bounded [5], if \(|\|f(s)\|| \leq K\), we have
\[
(Bs) \int_S |f| \, d\mu \leq (Bs) \int_S |f| \, d\mu \leq K \mu(S)
\]
In case when \(C\) is a subset of \(C\) such that \(\mu(C) < \delta = \frac{\varepsilon}{K}\) for every \(\varepsilon > 0\) we get
\[
(Bs) \int_C |f| \, d\mu < \varepsilon.
\]
(V) The inequality for the simple determinant functions \(|\|f_k\|| \leq |\|g_k\||\) a.a.k implies
\[
\int_S |f_k| \, d\mu \leq \int_S |g_k| \, d\mu \text{ a.a.k.}
\]
Isotonic property of integrals gives
\[
(Bs) \int_S |f| \, d\mu \leq (Bs) \int_S |g| \, d\mu
\]
(VI) If \(A_1\) and \(A_2\) are subset of \(\sigma\)-algebra \(A\) and \(A_1 \subset A_2\) then
\[
(Bs) \int_{A_1} |f| \, d\mu \leq (Bs) \int_{A_2} |f| \, d\mu
\]
Inequality follows by the property
\[
(Bs) \int_{A_1} |f| \, d\mu = (Bs) \int_{A_2} |f| \chi_{A_1} \, d\mu \leq (Bs) \int_{A_2} |f| \, d\mu
\]
(VII) Chebishev inequality
Let \(f_k\) and \(f\) be the st-integrable functions. If for every \(\varepsilon > 0\) we have \(|\|f_k(s) - f(s)\|| \geq \varepsilon\) for every \(s \in A_k\) where \(A_k \in A\) and \(A_k \subset A\) then from property(V) we have
\[
(Bs) \int_A |f_k - f| \, d\mu \geq (Bs) \int_{A_k} |f_k - f| \, d\mu \geq \varepsilon \mu(A_k)
\]
This inequality implies that for every \(\varepsilon\) and \(A\)
\[
\mu(\{x : |f_k - f| \geq \varepsilon\}) \leq \frac{1}{\varepsilon} (Bs) \int_A |f_k - f| \, d\mu
\]

4. Convergence theorems of statistically integral

**Theorem 15 (Fatou)**
Let \((f_k(s))\) be the sequence of st-measurable functions statistically convergent almost everywhere to the function \(f(s)\). If for a.a.k and every \(s \in S\) \(||f_k(s)|| \leq ||f_{k+1}(s)||\), then
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\[ st - \lim_{r \to S} \int_S \| f_r \| \, d\mu = (Bs) \int_S \| f(s) \| \, d\mu \]

**Proof.** We present every function \( \| f_k \| \) as a limes of no decreasing sequence of simple functions \( \| f'_k \| \) convergent to \( \| f_k \| \) a.a. k. For example as such sequence we can use

\[ f'_k = \begin{cases} h2^{-r} & \text{for } h2^{-r} \leq \| f_k(s) \| < (h + 1)2^{-r} \quad h = 0, 1, ..., 2^{2r} - 1 \\ 2^r & \text{for } \| f_k(s) \| \geq 2^r \end{cases} \]

Now, we construct the simple functions

\[ g^r(s) = \max \{ \| f'_1(s) \|, ..., \| f'_k(s) \| \} \]

We note that \( \| f'_k(s) \| \leq \| f_k(s) \| \) for every k and almost all r also \( \| f_k(s) \| \leq \| f(s) \| \) for all k \( < r \). We obtain the inequalities

(1) \( \| f'_k(s) \| \leq g^r(s) \leq \| f_k(s) \| \)

By the st-integral properties we have

(2) \( (Bs) \int_S \| f'_k(s) \| \, d\mu \leq (Bs) \int_S g^r(s) \, d\mu \leq (Bs) \int_S \| f_k(s) \| \, d\mu \)

By virtue of isotony and inequality (1) we get

\[ \| f_k(s) \| \leq st - \lim_{r \to S} g^r \leq \| f(s) \| . \]

The limes show that for almost all r the sequence \( (g^r(s)) \) is a nondecreasing sequence bounded by above from the function \( \| f(s) \| \). Taking in account that \( f_k(s) \to f(s) \) almost every \( s \in S \) and almost all \( m \) we get \( g^r(s) \to \| f(s) \| \) almost every \( s \) and almost all \( r \).

Hence the sequence \( (\| f'_k \|) \) is also sequence of simple functions by the definition of st-integral we write

\[ \int_S \| f'_k(s) \| \, d\mu \leq \int_S \| f(s) \| \, d\mu \leq st - \lim_{r \to S} \int_S f_r \, d\mu \]

so

\[ st - \lim_{r \to S} \int_S \| f_r \| \, d\mu = (Bs) \int_S \| f(s) \| \, d\mu . \]

**Theorem 16.** Let \( f(x) \) be the function with value in separable Banach space and st-measurable by a probability measure. If for almost all \( s \in S \) holds the inequality \( \| f(s) \| \leq g(s) \),

where \( g(s) \) is a function statistically integrable then the function \( f(s) \) is statistically integrable.
**Proof.** We denote \( f^m_k \) the sequence of simple determinant functions that converges statistically to the function \( f \). It holds
\[
\| f^m_k \| \leq \| f \| + \| f^m_k - f \| + \varepsilon
\]
\[
< \| f \| + \frac{1}{2} \| f^m_k \|
\]
so
\[
\| f^m_k \| \leq 2 \| f \| \leq 2g \text{ almost all } k \text{ and } m.
\]
and
\[
\| f^m_k(s) - f^m_N(s) \| \leq \| f^m_k(s) \| + \| f^m_N(s) \| < 4g(s)
\]
By the property (V) of st-integral, we have
\[
(Bs) \int S f^m_k(s) - f^m_N(s) d\mu \leq (Bs)4 \int g(s) d\mu
\]
a.a. \( k \).
If we take the measurable set \( C \) such that \( \mu(C) < \delta \) in virtue of property (IV) for the st-integral, we get
\[
\int_C g(s) d\mu < \varepsilon.
\]
So, we prove
\[
\int_C \| f^m_k - f^m_N \| d\mu < \varepsilon.
\]

Hence the sequence \( (f^m_k) \) converges almost everywhere on \( S \) and a.a.k to the function \( f \) which implies that measure of set \( B = \{ s : \| f^m_k(s) - f^m_N \| \geq \lambda, \lambda > 0 \} \) is zero.

Let \( S \setminus B \) be the set \( \{ s : \| f^m_k(s) - f^m_N \| < \lambda \} \). By the property (II) of st-integral
\[
\int_S \| f_k - f_N \| d\mu = \int_B \| f_k - f_N \| d\mu + \int_{S \setminus B} \| f_k - f_N \| d\mu
\]
\[
< \varepsilon + \lambda \cdot \mu(S)
\]
So, we have \( \text{st-lim} \int_S f_k(s) d\mu = 0 \).

The proof is completed if we substitute the function \( g(s) \) with \( ||f(s)|| \).

**Theorem 17. (Lebesgue)**

Let \( (f_k(s)) \) the sequence of st-measurable functions with value in separable Banach space convergent to the function \( f(x) \) almost everywhere and a.a.k. If for the functions \( f_k(s) \) holds a.a.k. the inequation \( ||f_k(s)|| \leq g(s) \), where \( g(s) \) is st-integrable function, then
\[
(Bs) \int f(s) d\mu = \text{st-} \lim \int f_k(s) d\mu.
\]
**Proof.** By the theorem 16, the functions $f_k$ are st-integrable. The inequality $\|f_k(x)\| \leq g(s)$ and convergence $f_k(s) \to f(s)$ almost everywhere and a.a.k implies inequality $\|f(s)\| \leq g(s)$ a.e. This means that the function $f(s)$ is st-integrable and except this the inequality $0 \leq \|f_k(s)-f(s)\| \leq 2g(s)$ holds a.e. This proves that the function $\|f_k(s)-f(s)\|$ is st-integrable a.e.

$$st \lim_{k} \int_S \| f_k(s) - f(s) \| \, d\mu = 0$$

Inequality

$$\| \int_S f_k(s) \, d\mu - \int_S f(s) \, d\mu \| \leq \int_S \| f_k(s) - f(s) \| \, d\mu$$

yields that sequence $((B_s)f_k(s))$ is st-convergent moreover uniformly to the st-integral $(B_s)f(s)$.

**References**


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