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Chaotic Evaluations in a Modified Coupled Logistic Type Predator-Prey Model

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Abstract

Regular and chaotic oscillations in a modified discrete two dimensional coupled predator-prey model, proposed recently, is re-investigated by observing the bifurcation diagram, calculating Lyapunov exponents and correlation dimensions for various orbits. The map evolves from one cycle to three cycles followed by period doubling and then to a chaotic regime and show bistability as its parameter λ varies, $0 < \lambda \leq 1.211$. Initial population size of the species and the value their coupling coefficients play crucial role for subsequent evolutionary phenomena of the system. Regular and chaotic evolutions depend completely on the coefficient ,λ, and on the initial prey and predator population. Lyapunov exponents and correlation dimensions provide true measure of chaos and also, to identify the chaotic orbits. Some recently discovered indicators, FLI, SALI and DLI have also been used to understand the nature of orbits. Mathematical analysis have been carried out to find stable and unstable fixed points and to obtain numerical value of the parameter λ for sensitive state of the system to initial conditions i.e. emergence of chaos. Bistability condition have been discussed for variation of λ values. Graphical representation of Lyapunov exponents and correlation dimensions provide better understanding of the nature of orbits which are further justified by the use of FLI, SALI and DLI.

Keywords: Lyapunov exponents, bistabiity,modified predator-prey interaction, complex patterns

1. Introduction

Competition between species for their survival under various environmental conditions, interactions among species and their relationship have been described recently through a number of research articles. Though not very realistic, the first mathematical approach for predator-prey model was initiated by Lotka (1925) and Volterra (1926) which now known as Lotka-Volterra model. A number of recent articles have been appeared on Lotka-Volterra model for different evolutionary conditions, e.g., Tedeschini-Lalli (1995), Li et al (2004), Son et al (2004), and drawn very interesting results. Logistic map and coupled logistic map have been used to describe population dynamics in many context e.g. Verhulst (1845), May (1976), Feigenbaum (1978) etc. This logistic map have large applications in different areas. López-Ruiz and Fourner-Prunaret, (2003- 2005), have interesting articles on predator-prey model for different choices of λ_n. In their paper, López-Ruiz and Fourner-Prunaret, (2005), proposed a predator-prey problem shown as a cubic discrete coupled logistic equation with assumption that the coupling depending on the population size of species and on a positive constant λ. They further assumed that, when each species living in isolation, the prey growth capacity is 4 times stronger than that of the predator.This constant itself depending on the prey reproduction rate as well as on the predator hunting strategy. Conditions of coexistence of predator and prey populations, extinction of ether one or both and their periodic, quasi-periodic and chaotic oscillations for parameter λ have been laid down. The modified coupled logistic model of predator-prey is written as follows:

$$\begin{aligned} x_{n+1} &= \lambda\,(3\,y_n + 1)\,x_n\,(1 - x_n) \\ y_{n+1} &= \lambda\,(-3\,x_n + 4)\,y_n\,(1 - y_n), \end{aligned} \tag{1.1}$$

where x_n, y_n, respectively, represents the predator and prey populations at generation n. Evolutionary dynamics of each species is similar to that of the logistic map. For a value of λ the above system have five fixed points. Stability analysis together with iterations, could decide the nature of orbits. Varying parameter λ from λ = 0 to λ = 1.21091, we can obtain interesting criteria of bifurcation leading to chaos.

The system (2.1) has five fixed points and for a particular λ value, their stability can be examined. For various range of values of λ, existence of attractors can be obtained by using the method of dynamical systems. Also, to observe regular and chaotic fluctuations, in addition to time series and phase plane plots, one should extend the work further to calculate largest Lyapunov exponents and FLI, SALI and DLI plots.

The objective of this study is to extend further study on system (1.1) and to search for periodic orbits as well as chaotic evolutions. We plot the bifurcation diagram of this map by varying λ, $0 \le \lambda \le 1.22$. This plot allow us to observe a complete scenario of evolution and to separate regular and chaotic zones of evolution. In the process we would like to obtain some periodic orbits also. Then, we proceed further to find the plots for largest Lyapunov exponents for chaotic fluctuation and then to obtain FLI SALI and DLI plots for regular as well as chaotic motions. For the case of chaos, we wish to calculate the correlation dimension also.

2. Bifurcation Analysis and Measure of Chaos

Investigating chaotic evolution in a ecological system is a challenging problem due to the its complex nature. The usual analysis through time series, phase plot, Poincaré map may not be enough to explain such complexity. A bifurcation diagram of the system certainly clarifies its evolution as parameters, (e.g., in our case λ in eqn. 1.1)), within the system changes in certain manner. We need some other elements to characterize deterministic chaos. Important among these are Lyapunov exponents and correlation dimension, Grassberger and Procaccia(1983). There may also be other Measures such as topological entropy etc. But in this study our concentration confined to above two only.

Bifurcation Diagrams:
Bifurcations in the system occurs during its evolution while changing the parameter λ. Here, we have varied the parameter λ in the renge $0 \le \lambda \le 1.22$ and drawn the bifurcation diagram for the model (1.1) and shown in Fig. 2.1.

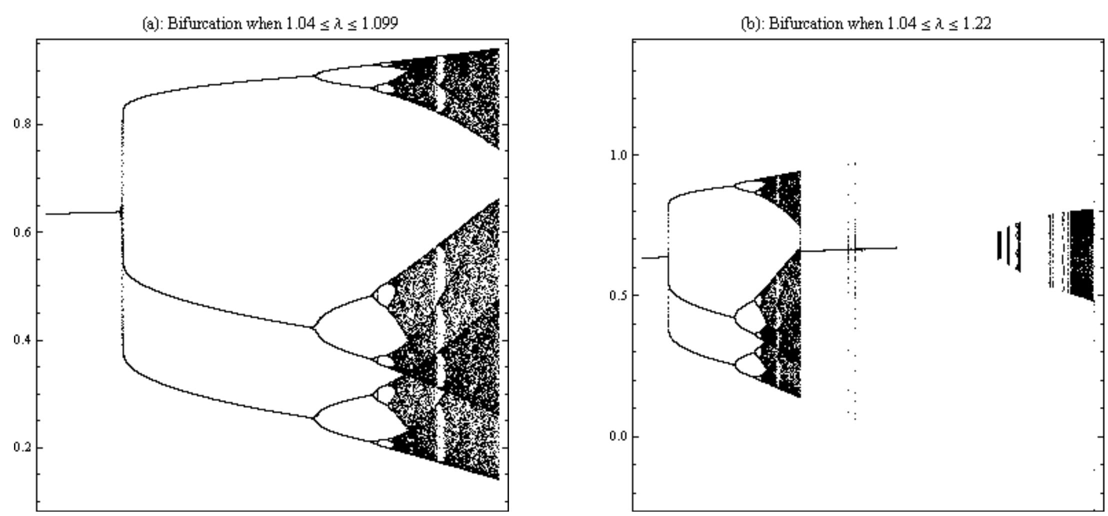

Fig.2.1: Bifurcation diagram of map (1.1) showing trifurcation of cycle one followed by period doubling and then chaos.
When λ exceeds, approximately, the value 1.092, one observe again a periodic window of period 9 and followed by chaos. Then, we see an one cycle and after discontinuity of the plot due to

occurrence of singularities in the system and can be seen from Fig.2.2. Finally, for range of values of λ, we observe a repeat of the bifurcation scenario, as in the second figure in Fig.2.1.

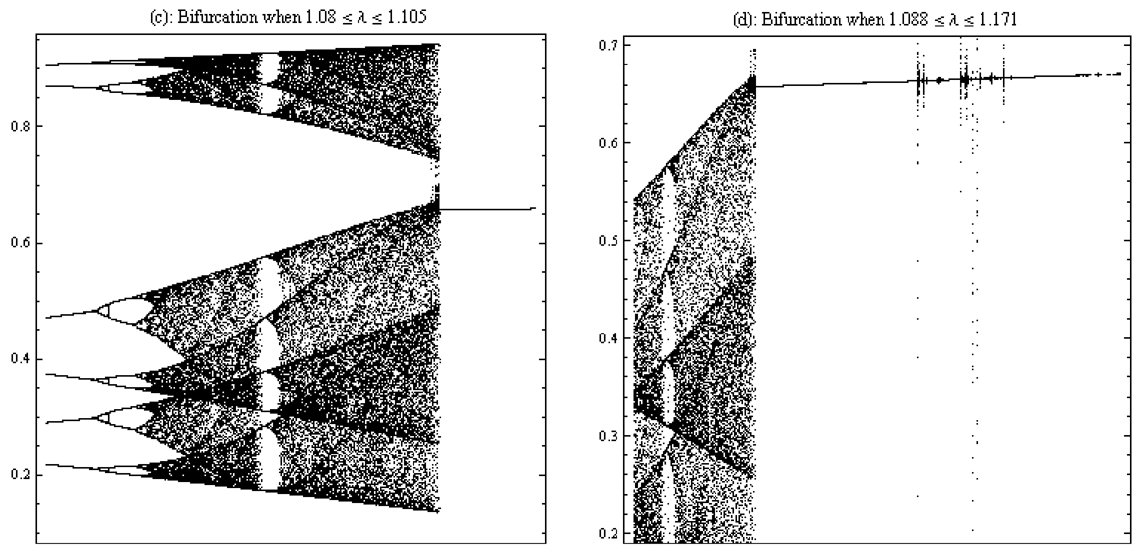

Fig.2.2: Bifurcations for $1.08 \leq \lambda \leq 1.105$ and $1.088 \leq \lambda \leq 1.171$

The above bifurcation figures clearly indicate the number of attractors and the type of bifurcations mentioned in the work of López-Ruiz and Fourner-Prunaret, (2005). The coexistence or extinct of one or both the species and the fluctuations of their numbers depend on range of values of parameter λ as well as on the initial numbers of both the specdies (e.g. x_0 and y_0).

Lyapunov Exponents (Lyapunov Numbers):

Lyapunov exponents are dynamical measure capable to characterize deterministic chaos in the system which features to the highly sensitive dependence on initial conditions. Actually it means the exponential divergence of orbits originated closely with very small difference in initial conditions. It is an important and effective element to identify regularity and chaos in the system and can be explained in the following ways:

Lyapunov Numbers : Chaos in a dynamical system is characterized by the exponential divergence of orbits originated closely. Such complexity of behavior in solution can be measured by a quantity called Lyapunov number. Lyapunov exponents are the measure of divergence of two orbits originated with slightly different initial conditions. For any one dimensional map defined in some interval (a, b),

$$x_{n+1} = f(x_n) \quad (2.1)$$

and its two orbits starting at x_0 and $x_0 \pm \delta_0$, where δ_0 is very small , expanding $f(x_0 + \delta_0)$ by taylors series, then after one iteration the distance between the orbits be given by

$$\delta_1 = I\, f'(x_0)\, I/ \delta_0 = M_0\, \delta_0 \quad (2.2)$$

M_o is known as first step magnification factor. Similarly, at the second iteration, the distance between the orbits can be written as

$$\delta_2 = I\, f'(x_1)\, I/ \delta_1 = M_1\, \delta_1 = M_1\, M_0\, \delta_0 \quad (2.3)$$

Continuing in this manner, separation between the orbits at n^{th} iteration is

$$\delta_n = I\, f'(x_{n-1})\, I/ \delta_{n-1} = M_{n-1}\, \delta_{n-1} = M_{n-1}\, M_{n-2} \ldots M_0\, \delta_0 \quad (2.4)$$

The product $M_0\, M_1\, M_2 \ldots.. M_{n-1}$ is the accumulation of magnification factors, so it is meaningful to consider an average of it. The most convenient is the geometric average

$$(M_0\, M_1\, M_2 \ldots.. M_{n-1})^{1/n}$$

Taking log, one obtains the arithmetic average

$$\lambda = \ln (M_0\, M_1\, M_2 \ldots.. M_{n-1})^{1/n} = 1/n\, (\ln M_o + \ln M_1 + \ln M_2 + \ldots + \ln M_{n-1})$$

$$= 1/n\, (\ln (I\, f'(x_o)\, I) + \ln (I\, f'(x_1)\, I) + \ln (I\, f'(x_2)\, I) + \ldots + \ln (I\, f'(x_{n-1})\, I)) \quad (2.5)$$

Then, the condition of stability of a implies:

If average magnification is less than 1, the orbit is stable and if it is greater than 1 the orbit is unstable, i.e. $\lambda < 0$ => stable orbit and $\lambda > 0$ => unstable orbit. For accurate result, one should take the iterations n as large as possibleThis leads to the following definition of Lyapunov exponents:

Def. 1: **Lyapunov exponents** of a smooth map f on R with x_0 as initial point be defined as

$$\lambda(x_0) = \lim_{n \to \infty} 1/n\, [\ln (I\, f'(x_0)\, I) + \ln (I\, f'(x_1)\, I) + \ln (I\, f'(x_2)\, I) + \ldots + \ln (I\, f'(x_{n-1})\, I)]$$

provided the limit exists. Lyapunov number is the exponent of Lyapunov exponent and is given by

$$L(x_o) = e^{\lambda(x_0)} \qquad (2.6)$$

Def. 2: A bounded orbit $\{x_0, x_1, x_2, x_3, \ldots, x_n\}$ of the map f on R is called chaotic if following conditions are satisfied:

(a) $\{x_0, x_1, x_2, x_3, \ldots, x_n\}$ is not asymptotically periodic
(b) No $\lambda(x_0)$ is exactly equal to zero , and
(c) $\lambda(x_0) > 0$ or equivalently, $L(x_o) > 1$.

From above definition, a clear interpretation for Lyapunov exponent is given as: it is the measure of loss of information during the process of iteration.

For higher dimensional system, we can generalize the above one dimensional case to higher dimension and obtain

$$\lambda(X_0, U_0) = \lim_{n \to \infty} \frac{1}{n} \log \left\| \coprod_{t=0}^{n-1} J(X_t) U_0 \right\|, \qquad (2.7)$$

and

$$\left\| X_n - Y_n \right\| \approx e^{\lambda(X_0, U_0) n},$$

where $X \in P^{n,}$ $F: P^n \to P^n$, $U_0 = X_0 - Y_0$ and J is the Jacobian matrix of map F.

Quantitatively, two trajectories in phase space with initial separation δx0 diverge (provided that the divergence (can be treated within the linearized approximation)

$$|\delta x(t)| \approx e^{\lambda t} |\delta x(0)| \qquad (2.8)$$

where $\lambda > 0$ is the Lyapunov exponent.

The system described by the map f be *regular* as long as $\lambda \le 0$ and *chaotic* when $\lambda > 0$.

For same parameter value but different initial conditions results could be very different. For, take $\lambda = 1.095$ and initial condition $(x_0, y_0) = (0.4, 0.3)$, the evolution becomes chaotic and in this case the largest value of Lyapunov exponents can be obtained as 0.377034 whereas for same value of λ but $(x_0, y_0) = (0.6, 0.6)$, the system becomes regular and minimum value of Lyapunov exponents be equal to -0.10699. Plots of Lyapunov exponents for these two cases are shown in Fig. 2.3. For a value of λ greater than 1.09969, one can observe a type of constant equilibrium for populations, (left figure in Fig.2.2), and no chaotic band. For $\lambda > 1.1759$, one observes Hopf bifurcation emerging in the form of certain invariant curves. This indicated various states of population oscillations. Further increasing λ resulting in overflows of iterations. This amounts to certain uncertainty situation or crash in the system. There may be a catastrophic situation leading to extingtion of both the species.

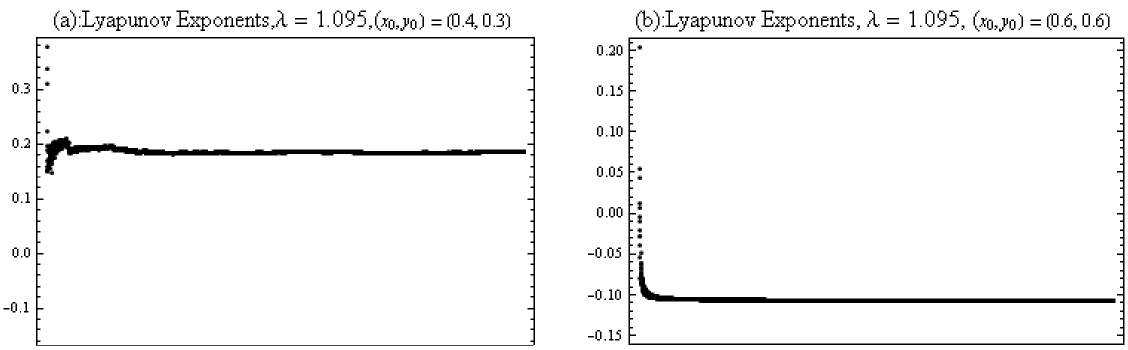

Fig. 2.3: Plot of largest Lyapunov exponent for $\lambda = 1.095$ and (x_0 , y_0) are (0.4, 0.3) in figure (a) and (0.6, 0.6) in figure (b).

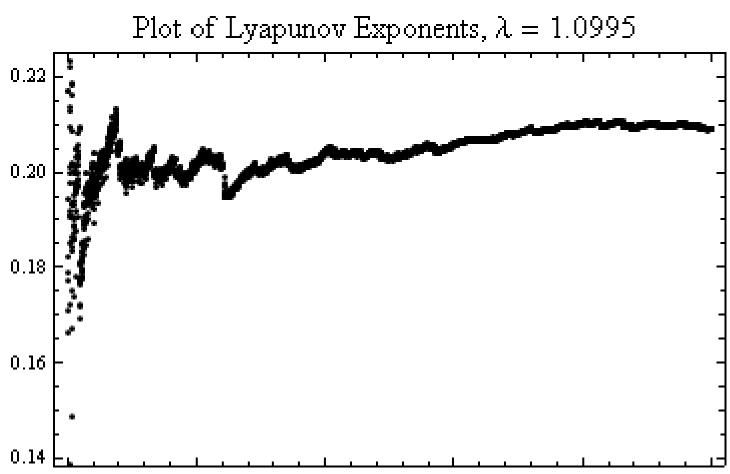

Fig.2.4: Plot of Lyapunov exponents for $\lambda = 1.0995$ and $(x_0 , y_0) = (0.4, 0.3)$. A largest Lyapunov exponent is 0.381135 in this case and its average is 0.204675.

Aperfect equilibrium state reached for approximately λ = 1.2101 when both when initially both the species are equal in number 0.5, i.e. x = 0.5 and y = 0.5. A phase diagram for this case appears to be a limit cycle and, except few initial iterations, Lyapunov exponents have very small negative values near to zero. This is a case of neutral stabilitry and coexistence occur for boyth species. Both plots are shown in Fig. 2.5.

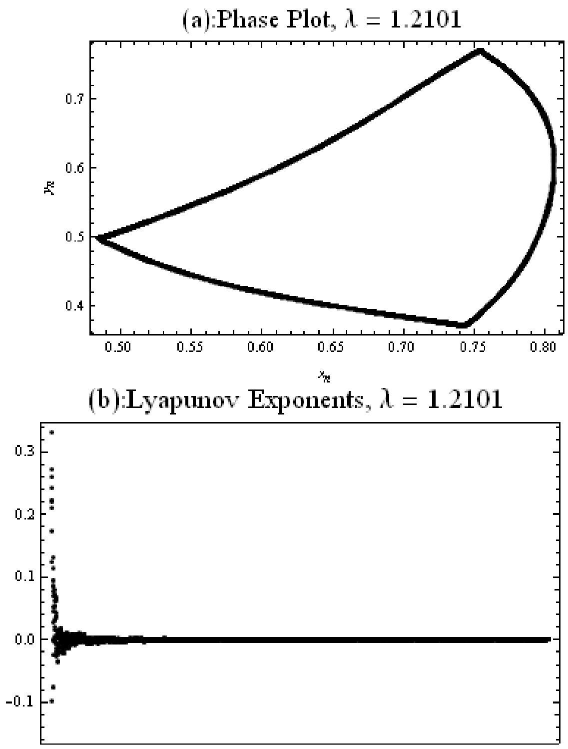

Fig.2.5: Phase plot and corresponding plot for Lyapunov exponents for λ = 1.2101 and x_0 = 0.5 and y_0 = 0.5. Phase plot is a limit cycle and Lyapunov exponents approximately zero.

Correlation Dimension

Chaos appears during evolution in nonlinear systems in the form of *strange attractor* which has fractal properties. Also we have the notion of weak and strong chaos. Such notion may be justified by some measure of chaos. Correlation dimension gives a *measure of dimensionality* of the chaotic set. Being one of the characteristic invariants of nonlinear system dynamics, the correlation dimension actually gives a measure of complexity for the emerging chaotic attractor of the system. By calculating this dimension one could be in a position do explain the nature of chaos appearing in the system. To determine correlation dimension we use statistical method described in the books by Nagashima and Baba (2005) and Martelli (1999). The methods described in these books are very practical and efficient compared to many other methods, like box counting etc. The procedure to obtain correlation dimension follows the following steps, Martelli (1999):

Consider an orbit $O(\mathbf{x}_1) = \{\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, \mathbf{x_4}, \}$, of a map f: U → U, where U is an open bounded set in P^n. To compute correlation dimension of $O(x_1)$, for a given positive real number r, we form the correlation integral, Grassberger and Procaccia (1983),

$$C(r) = \lim_{n\to\infty} \frac{1}{n(n-1)} \sum_{i\neq j}^{n} H\left(r - \left\| x_i - x_j \right\| \right), \tag{2.9}$$

where

$$H(x) = \begin{cases} 0, & x<0 \\ 1, & x\geq 0 \end{cases},$$

is the unit-step function, (Heaviside function). The summation indicates the the number of pairs of vectors closer to r when $1 \leq i, j \leq n$ and $i \neq j$. C(r) measures the density of pair of distinct vectors $\mathbf{x_i}$ and $\mathbf{x_j}$ that are closer to r.

The correlation dimension D_c of $O(\mathbf{x_1})$ is defined as

$$D_c = \lim_{r\to 0} \frac{\log C(r)}{\log r} \tag{2.10}$$

To obtain D_c, log C(r)/logr is plotted against r, (the plot may be called a plot of correlation curve), and then we use the least square method to find a straight line fitted to this curve. The y-intercept of this straight line provides the value of the correlation dimension D_c. For different orbits, i.e. the orbits due to different initial conditions, and different parameter values λ, one gets different correlation dimensions signifying the order of chaotic orbits, (how weak or how strong the chaotic orbit is).

For λ = 1.099, correlation dimension for the orbit of (0.4, 0.3) is D_C = 0.985113 whereas for orbit of (0.55, 0.5), correlation dimension is D_C = 0.000736621. The corresponding correlation curves are given in Fig.2.6

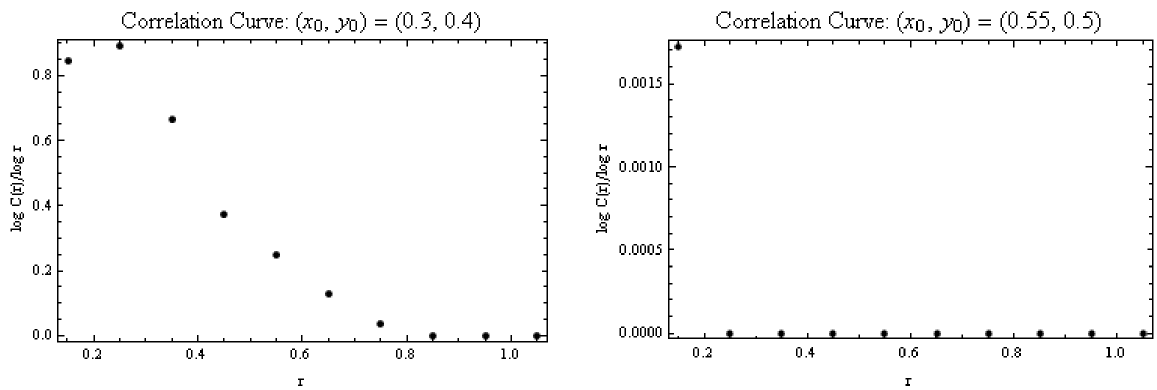

Fig.2.6: Correlation curves for orbits of (0.3, 0.4) and (0.55, 0.5) when λ = 1.099.

The equations of straight lines, obtained with least square fitting method, for above correlation curves respectively, be given by

$$y = 0.985113 - 1.1098\ x$$

and

$$y = 0.000736621 - 0.000940368.$$

It is clear from the second equation that the correlation dimension of the orbit of (0.55, 0.5) when λ = 1.099 is almost zero. We have continued our calculation for correlation dimensions for various orbits and with different parameter values λ and shown in Table 1.

Table: 1: Calculating Correlation dimension for $1.09 \leq \lambda \leq 1.22$.

Observation No.	Value of Parameter λ	Initial Values of (x_0, y_0)	Correlation Dimension D_C
1	1.09	(0.55, 0.5)	0.00036801
2	1.095	(0.4, 0.3)	0.835527
3	1.095	(0.6, 0.6)	0.000921154
4	1.095	(0.55, 0.5)	0.00055224
5	1.099	(0.55, 0.5)	0.000736621
6	1.099	(0.4, 0.3)	0.985113
7	1.099965	(0.3, 0.6)	1.21806
8	1.099965	(0.55, 0.5)	0.000921154
9	1.1	(0.5, 0.4)	0.969251
10	1.1	(0.55, 0.5)	0.000921154

11	1.1	(0.5, 0.5)	0.0105549
12	1.1	(0.6, 0.5)	0.0000
13	1.17565	(0.5, 0.5)	0.1293
14	1.17565	(0.55,0.44)	3.35727
15	1.2	(0.4, 0.4)	4.10614
16	1.2	(0.5, 0.5)	0.350668
17	1.2	(0.55, 0.5)	0.29756
18	1.211	(0.0.4, 0.6)	0.551895
19	1.211	(0.6, 0.5)	0.383504
20	1.211	(0.55, 0.5)	0.49418
21	1.211	(0.5, 0.5)	0.572222
22	1.211	(0.5, 0.55)	0.383504
23	1.22	(0.6, 0.5)	1.98922
24	1.22	(0.55, 0.5)	2.04899
25	1.2101	(0.5, 0.5)	0.496059
26	1.2101	(0.55, 0.45)	0.497083

A discussion for numerical results obtained in Table 1will be followed in the last section. However, an important general type of characteristic behavior has been observed from the numerical investigation, shown in Table 1, is that some orbits (0.5, 0.5), (0.55, 0.5), (0.6, 0.5) are less chaotic for $1.09 \leq \lambda \leq 1.211$. This means, when the population size of species are near to 0.5, the possibility of their coexistence becomes more certain. But when their population size differ significantly, such coexistence become unpredictable because of their chaotic evolution. In some cases like this iterations becomes overflows and numerical prediction would be impossible.

3. Application of Indicators FLI, SALI and DLI

For clear identification of regular and chaotic orbits, some novel indicators, known as the Fast Lyapunov Indicators (FLI), the Smaller Alignment Indices (SALI) and the Dynamic Lyapunov Indicators (DLI), have been defined. These indicators could made studies on evolving systems more precise and meaningful. These have been discovered by various authors while studying discrete and continuous models of their interest. The concept of Fast Lyapunov Indicator (FLI) was introduced by Froeschle et al (1997) and applied again in their work Lega and Froeschle (2002) and also, by Saha et (2006); Smaller Alignment Indices (SALI) introduced by Skokos (2001) and again in the work Skokos et al (2004). More recently, the indicator named as

Dynamic Lyapunov Indicator (DLI) has been introduced by Saha and Budhraja (2007) which again applied in the work by Budhraja (2008). It has been observed that DLI gives very clear indication of ordered and chaotic motion whenever applied. Definitions of FLI, SALI and DLI can be obtained from the articles of their respective introducers. However, we must keep in mind the properties of these indicators as follows:

FLI*'s increase exponentially for chaotic orbits and linearly for regular orbits*.
SALI's *fluctuates around a non-zero value for ordered orbits while it tends to zero for chaotic orbits*.
DLI's , *form a definite pattern, then the motion is regular and if they are distributed randomly, (with no definite pattern), then the motion is chaotic*.

Numerical Calculations for FLI, SALI and DLI:
For our system (1.1) we have selected few orbits which evolve regularly or chaotically for some particular value of λ. For these orbits we made numerical calculations and draw graphs for above indicators.

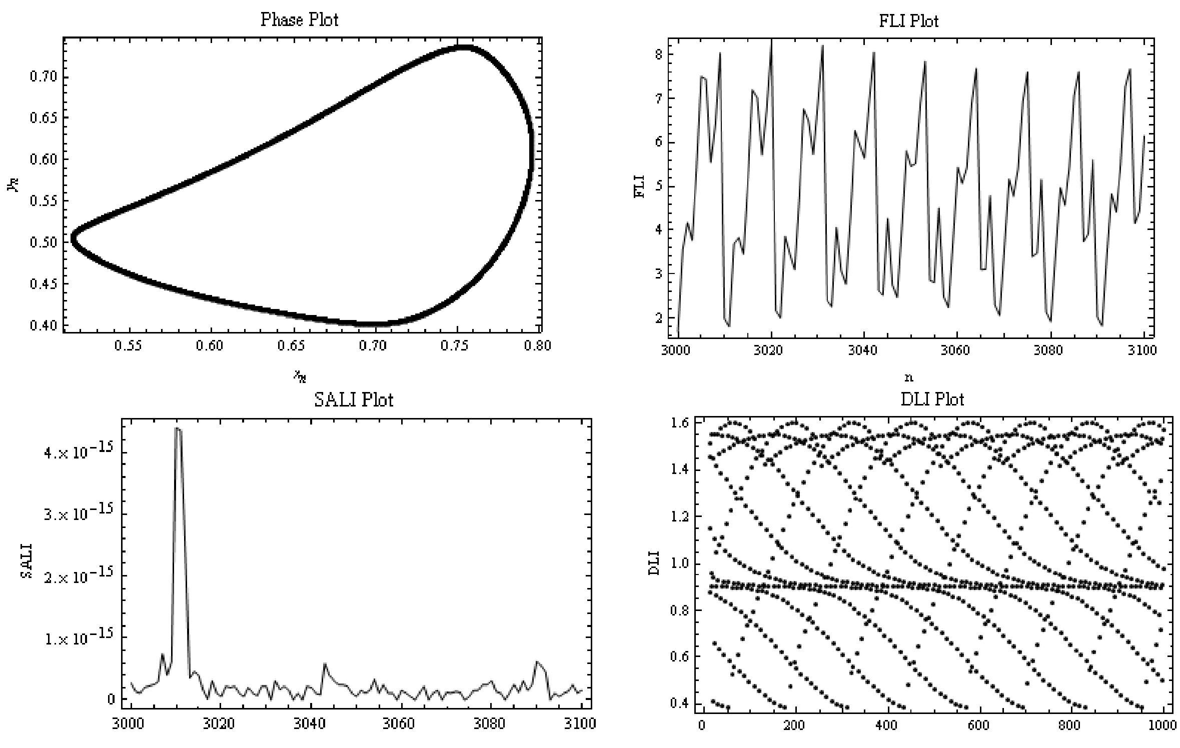

Fig.3.1: Phase plot and FLI, SALI , DLI plots of a regular orbit (0.5, 0.5) for λ = 1.2. The last figure shows the emergence of a pattern in DLI plot.

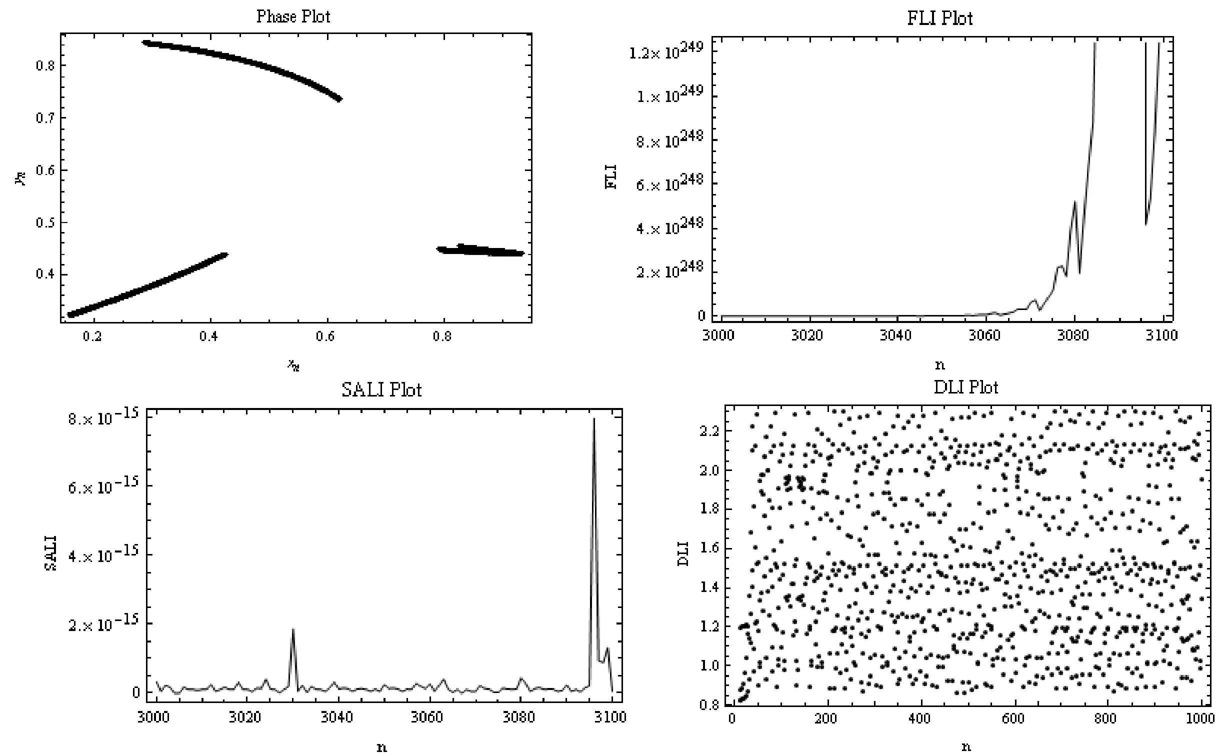

Fig.3.2: Phase plot and FLI, SALI , DLI plots of a chaotic orbit (0.4, 0.3) for λ = 1.095. The last figure shows randomly distributed points in the DLI plot.

4. Discussions

Bifurcation diagrams, Fig. 2.1 and Fig.2.2, give clear indication regarding evolutionary scenario of the system. These figures show the appearance of bistability. Evolution is here of very specific kind. We see first a cycle one which trifurcates as λ increases and then we observe doubling phenomena of three cycles for certain range of the parameter. Finally a chaotic state. From this, we again see one cycle. Further increase in λ results in some kind of brusts in numerical calculations.

Lyapunov exponents and correlation dimensions are measure of chaos and their numerical values, when calculated for any orbit, gives clear idea of regular or chaotic nature of the orbit. As shown in Fig.2.3, Lyapunov exponents calculated for regular orbit are negative and those for chaotic orbit are positive.

Numerical calculations of correlation dimensions show some interesting results. Orbits are regular when initial values of species are nearby 0.5 for possible range of values of λ. For λ = 1.1, orbits of (0.5, 0.5), (0.55, 0.5), (0.6, 0.5) are found regular whereas for same λ, the orbits (0.5, 4) obtained to be chaotic. Similar type of orbits also observed for other value of parameter λ. This provides an indication that the possibility of co-existence may happen if both the population starts nearby 0.5. This again leads to the situation of bistability.

In Fig.3.1 and 3.2 we have plotted FLI, SALI and DLI, respectively, for a regular and a chaotic orbit. No linear increment observed in FLI for the our model; hoever, it shows nearly exponential increase has been observed for chaotic case. SALI seems to be working for ordered as well for chaotic orbit. In case of DLI, one can see it is working perfectly as per its definition. DLI display a pattern for ordered orbit and randomly distributed points for chaotic orbit. As tested several times in past on various discrete systems, it reflects that DLI could be reliable indicator for distinguishing regularity and chaos. In a recent article Deleanu (2011) has applied DLI in a number of discrete systems and referred this indicator as a practical tool to distinguish ordered and chaotic orbits in dynamical systems.

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