On the Class of New Better than Used of Life Distributions

Zohdy M. Nofal
Department of Statistics
Mathematics and Insurance Faculty of Commerce
Benha University, Egypt
dr_znofal@hotmail.com

Abstract
Some new results about NBU(3) class of life distributions are obtained. Closure of the NBU(3) class under convolution are given. Shock models where shocks are arriving according to homogeneous are established.

Keywords: NBU(3), convolution, shock models, life distributions

1. Introduction
Certain classes of life distributions and their variations have been introduced in reliability, the applications of these classes of life distribution can be seen in engineering, social, biological science, maintenance and biometrics. Therefore, statisticians and reliability analysts have shown a growing interest in modeling survival data using classifications of life distributions based on some aspects of aging (see for example Barlow and Proschan (1981)). Among the most well known families of life distributions are the classes of increasing failure rate (IFR), increasing failure rate average (IFRA), new better than used (NBU), new better
than used in expectation (NBUE), and harmonic new better than used in expectation (HNBUE). For more details, readers are referred to Barlow and Proschan (1981), Hendi and Mashhour (1993), Chen (1994), Belzunce et al. (2001), Li et al. (2000), Belzunce et al. (2001), Li and Kochar (2001), Franco et al. (2001), Nanda et al. (2003), Li and Zou (2004), and Ahmad et al. (2006) among others.

In this section, we review some of the aging criteria and their relationships. We also describe how the aging properties of the original distribution are transformed into the aging properties of the residual life.

Let $X$ be a non-negative random variable representing life of a component. Let $F$ be the cumulative distribution function of $X$ and $H(x)$ be the reliability function or the survival function of $X$.

Then $H(x) = P(X > x)$ is the survival function of a unit of age $x$.

Evidently, any study of the phenomenon of aging should be on $F$ and functions related to this. Thus

1. $F$ is said to be $PF_2$ if $\ln f(x)$ is concave, where $f(\cdot)$ is the density corresponding to $F(\cdot)$.
2. $F$ is said to have increasing (decreasing) failure rate, $IFR(DFR)$ if $F_t(x) = F(x + t)/F(t)$ is decreasing (increasing) in $t$. If $F$ is absolutely continuous with density $f$, then $F$ is in the $IFR(DFR)$ class if $r_F(t) = f(t)/F(t)$ is increasing (decreasing) in $t$.
3. $F$ is said to have increasing (decreasing) failure rate average, $IFR(DFR)$ if $\int_0^t r_F(x)dx/t$ is increasing (decreasing).
4. $F$ is said to have new better (worse) than used, NBU (NWU) if $F_t(x) \leq (\geq) F(t)$ for $x \geq 0, t \geq 0$.
5. $F$ is said to have increasing (decreasing) mean residual life, DMRL(IMRL) if the mean residual life $\mu_F(t) = \int_t^\infty F(x)dx/F(t)$ is decreasing (increasing) assuming that the mean $\mu_F(0)$ exists.
6. $F$ is said to have new better (worse) than used in expectation, NBUE(NWUE) if $\mu_F(t) \leq (\geq) \mu_F(0)$ for all $t \geq 0$.

The chain of implications between these classes of distributions is

\[
\text{IFRA} \Rightarrow \text{NBU} \Rightarrow \text{NBUC} \Rightarrow \text{NBUCA}
\]

\[
\uparrow \quad \downarrow
\]

\[
PF_2 \Rightarrow IFR \Rightarrow DMRL \Rightarrow NBUE \Rightarrow HNBUE \Rightarrow GHNBUE
\]

\[
\uparrow
\]

\[
\text{NBAFR} \Leftarrow \text{NBUFR} \Leftarrow \text{NBU} \Rightarrow \text{NBUA}
\]

Li and Kochar (2001) studied some properties of class NBU(2). The main theme of this is to further investigate the class. In section 2 definitions and some basic results are introduced. Section 3 includes closure properties under convolution and homogeneous Poisson shock model.

2. Definitions and Basic Results

**Definition 2.1.** A life distributions $F$ (i.e. $F(0^-)=0$) is called new better than used in a third order if

$$
\int_0^\infty \int_0^u \overline{F}(x+t)dtdu \leq \overline{F}(x)\int_0^\infty \int_0^u \overline{F}(t)dtdu
$$

(2.1)

**Definition 2.2.** A discrete distribution $H_g$ is said to be discrete NBU(3) if,

$$
\sum_{i=0}^{\infty} \sum_{j=0}^{k} \overline{P}(i+j) \leq \sum_{i=0}^{\infty} \sum_{j=0}^{k} \overline{P}(i)\overline{P}(j)
$$

(2.3)

3. The Closure Properties

3.1. Convolution

As an important reliability operation, convolution of life distribution of a certain class is often paid much attention. The closure properties of IFR, NBU, NBUE, and IFRA can be found in Barlow and Proschan (1981). The closure properties of class NBU(2) were pointed out in Hu and Xie (2002). In the next theorem we establish the closure property of the NBU(3) class under the convolution operation.

**Theorem 3.1**

Suppose that $F_1$ and $F_2$ are two independent NBU(3) life distributions. Then their convolution is also NBU(3)

**Proof**

$$
\overline{F}(x+t)=\int_0^\infty \overline{F_1}(x+t-z)dF_2(z)
$$

(3.1)
By integrating both sides twice first with respect to $t$, second with respect to $x$ then,

\[
\int_0^\infty \int_0^\infty \overline{F}(x+t)dt \, du = \int_0^\infty \int_0^\infty \overline{F}(x+t-z) \, d\overline{F}(z) \, dt \, du
\]

\[
= \int_0^\infty \int_0^\infty \overline{F}(x+t-z) \, dt \, du \overline{F}(z),
\]

but

\[
\int_0^\infty \int_0^\infty \overline{F}(x+t-z) \, dt \, du \leq \overline{F}(x) \int_0^\infty \overline{F}(t-z) \, dt \, du.
\]

Then,

\[
\int_0^\infty \int_0^\infty \overline{F}(x+t) \, dt \, du = \int_0^\infty \overline{F}(x) \int_0^\infty \overline{F}(t-z) \, dt \, du \overline{F}(z)
\]

\[
= \overline{F}(x) \int_0^\infty \int_0^\infty \overline{F}(t-z) \, dt \, du
\]

\[
= \overline{F}(x) \int_0^\infty \overline{F}(t) \, dt \, du
\]

**Theorem 3.2**

Suppose that $F_1$ and $F_2$ are two independent discrete NBU(3) life distributions. Then their convolution is also NBU(3).

**Proof**

\[
\overline{P}(i + j) \leq \sum_{z=0}^\infty \overline{P}_1(i + j - z) \, p_2(z) \quad \forall \; j = 0, 1, \ldots
\]

(3.2)

By summation respect to $x$, then

\[
\sum_{i=0}^\infty \sum_{j=0}^k \overline{P}(i + j) \leq \sum_{i=0}^\infty \sum_{j=0}^k \sum_{z=0}^\infty \overline{P}_1(i + j - z) \, p_2(z) \quad \forall \; j = 0, 1, \ldots
\]

\[
= \sum_{z=0}^\infty \sum_{i=0}^\infty \sum_{j=0}^k \overline{P}_1(i + j - z) \, p_2(z)
\]

\[
= \sum_{z=0}^\infty p_2(z) \sum_{i=0}^\infty \sum_{j=0}^k \overline{P}_1(i + j - z)
\]
3.2. Shock models

Suppose that a device is subjected to shocks occurring randomly as events in a Poisson process with constant intensity \( \lambda \). Suppose further that the device has probability \( \bar{P}_k \) of surviving the first \( K \) shocks. Then the survival function of the device is given

\[
\bar{H}(t) = \sum_{k=0}^{\infty} \bar{P}_k \frac{(\lambda t)^k}{k!} e^{-\lambda t}
\]  

(3.3)

For the discrete distribution \( \{\bar{P}_k, k \in N, \} \), it is well known that properties of \( \bar{P}_k \) are reflected in the corresponding properties of the continuous life distribution \( H(t) \). This is shown by Esary et al. (1973) for IFR, IFRA, DMRL, NBU and NBUE classes, Klefsjo (1981) for HNBUE and Abouammoh and Ahmed (1988) for NBUF

Definition 3.3

A discrete distribution or its survival probability function \( \{\bar{P}_k\}_{k=0}^{\infty} \) with finite mean \( m = \sum_{k=0}^{\infty} \bar{P}_k \) is called discrete NBU(3) if

\[
\sum_{i=0}^{\infty} \sum_{j=0}^{k} \bar{P}(i+j) \leq \sum_{i=0}^{\infty} \sum_{j=0}^{k} \bar{P}(i) \bar{P}(j) \quad \text{for all } j = 0, 1, ...
\]  

(3.4)
Theorem 3.4

The survival function $H(t)$ in (3.3) is NBU(3) if and only if $\{\bar{P}_k\}_{k=0}^\infty$ has the discrete NBU(3) property

Proof

Let $\bar{P}_k$ be the probability that the device survives the first $K$ shocks, where

$1 = \bar{P}_0 \geq \bar{P}_1 \geq \cdots$  The survival function is

$$\int_0^\infty \bar{H}(x+t) dtdu = \int_0^\infty \sum_{k=0}^\infty \bar{P}_k \frac{e^{-\lambda(x+t)} - \lambda(t+x)^k}{k!} dtdu$$

$$= \int_0^\infty \sum_{k=0}^\infty \bar{P}_k \sum_{j=0}^k (\lambda t)^j (\lambda x)^{k-j} \frac{e^{-\lambda(x+t)}}{k!} dtdu$$

$$= \int_0^\infty \sum_{k=0}^\infty \bar{P}_k \sum_{j=0}^k \frac{e^{-\lambda x} e^{-\lambda t}}{j!(k-j)!} dtdu$$

We know that

$$\sum_{i=0}^\infty \sum_{j=0}^\infty \bar{P}_{i+j} \leq \bar{P}_i \sum_{m=0}^\infty \sum_{j=0}^m \bar{P}_{j}$$

, then
\[
\int_0^\infty \int_0^\infty H(x+t)dtdu \leq \sum_{i=0}^{\infty} e^{-\lambda x} \frac{(-\lambda x)^i}{i!} F(i) \sum_{m=0}^{\infty} \sum_{j=0}^{m} \bar{P}_j \sum_{n=0}^{\infty} \frac{(\lambda y)^n}{n!} e^{-\lambda y} \\
\leq \bar{H}(x) \sum_{m=0}^{\infty} \sum_{j=0}^{m} \bar{P}_j \frac{1}{\lambda^2} \sum_{n=0}^{\infty} \frac{(\lambda y)^n}{n!} e^{-\lambda y} \\
= \bar{H}(x) \sum_{m=0}^{\infty} \sum_{j=0}^{m} \bar{P}_j \frac{1}{\lambda} \int_0^\infty \frac{\lambda u)^{m+1}}{(m+1)!} e^{-\lambda u} du \\
= \bar{H}(x) \int_0^\infty \sum_{j=0}^{\infty} \bar{P}_j \frac{1}{\lambda} \sum_{m=j}^{\infty} \frac{(\lambda u)^m}{(m)!} e^{-\lambda u} du \\
= \bar{H}(x) \int_0^\infty \left[ \sum_{j=0}^{\infty} \bar{P}_j \int_0^u \frac{\lambda t)^j}{(j)!} e^{-\lambda t} dt \right] du \\
= \bar{H}(x) \int_0^\infty \left[ \int_0^u \sum_{j=0}^{\infty} \bar{P}_j \frac{(\lambda t)^j}{(j)!} e^{-\lambda t} dt \right] du \\
= \bar{H}(x) \int_0^\infty \int_0^u \bar{H}(t)dtdu
\]

References


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