Isolated Points of Spectrum for

Quasi - * - Class A Operators

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Abstract. Let $T$ be a Quasi - $*$ - class $A$ operator on a complex Hilbert space $\mathcal{H}$ if $T^* (|T|^2 - |T^*|^2) T \geq 0$. In this paper, we prove that if $E$ is the Riesz idempotent for a non-zero isolated point $\lambda$ of the spectrum of $T \in$ Quasi - $*$ - class $A$ operator, then $E$ is self-adjoint and $EH = \ker(T - \lambda) = \ker(T - \lambda)^*$. We will also prove a necessary and sufficient condition for $T \otimes S$ to be quasi - $*$ - class $A$ where $T$ and $S$ are both non-zero operators.

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1. Introduction

Let $B(\mathcal{H})$ denote the algebra of all bounded linear operators acting on an infinite dimensional separable Hilbert space $\mathcal{H}$. For a positive operators $A$ and $B$, write $A \geq B$ if $A - B \geq 0$. If $A$ and $B$ are invertible and positive operators, it is well known that $A \geq B$ implies that $\log A \geq \log B$. However [2], $\log A \geq \log B$ does not necessarily imply $A \geq B$. A result due to Ando [5] states that for invertible positive operators $A$ and $B$, $\log A \geq \log B$ if and only if $A^r \geq (A^2 B^r A^2)^{\frac{1}{r}}$ for all $r \geq 0$. For an operator $T$, let $U|T|$ denote the polar decomposition of $T$, where $U$ is a partially isometric operator, $|T|$ is a positive square root of $T^* T$ and $\ker(T) = \ker(U) = \ker(|T|)$, where $\ker(S)$ denotes the kernel of operator $S$.

An operator $T \in B(\mathcal{H})$ is positive, $T \geq 0$, if $(Tx, x) \geq 0$ for all $x \in \mathcal{H}$, and posinormal if there exists a positive $\lambda \in B(\mathcal{H})$ such that $TT^* = T^* \lambda T$. Here $\lambda$ is called interrupter of $T$. In other words, an operator $T$ is called posinormal if $TT^* \leq c^2 T^* T$, where $T^*$ is the adjoint of $T$ and $c > 0$ [7]. An operator $T$ is said to be heminormal if $T$ is hyponormal and $T^* T$ commutes with $TT^*$. An operator $T$ is said to be $p$ - posinormal if $(TT^*)^p \leq c^2 (T^* T)^p$ for some $c > 0$. It is clear that 1 - posinormal is posinormal. An operator $T$ is said to be $p$ - hyponormal, for $p \in (0, 1)$, if $(T^* T)^p \geq (TT^*)^p$. An 1 -
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hyponormal operator is hyponormal which has been studied by many authors and it is known that hyponormal operators have many interesting properties similar to those of normal operators [19]. Furuta et al [9] have characterized class $A$ operator as follows. An operator $T$ belongs to class $A$ if and only if $(T^*|T|^2)^{\frac{1}{2}} \geq T^*T$.

An operator $T$ is said to be paranormal if $||T^2x|| \geq ||Tx||^2$ and $\ast$ - paranormal if $||T^2x|| \geq ||T^*x||^2$ for all unit vector $x \in \mathcal{H}$. Recently, B. P. Duggal et al [8] have considered the new class of operators : An operator $T \in B(\mathcal{H})$ belongs to $\ast$ - class $A$ if $|T^2| \geq |T^*|^2$. The authors of [14] have extended $\ast$ - class $A$ operators to quasi- $\ast$ - class $A$ operators. An operator $T \in B(\mathcal{H})$ is said to be quasi- $\ast$ - class $A$ if $T^*|T|^2T \geq T^*|T^*|^2T$ and quasi- $\ast$ - paranormal if $||T^*Tx||^2 \leq ||T^3x||||Tx||$ for all $x \in \mathcal{H}$. An operator $T$ is said to be Quasi- $\ast$ - class $A$ [12] operator on a complex Hilbert space $\mathcal{H}$ if $T^*|T^2| - |T^*|^2T \geq 0$.

As a further generalization, Mecheri [13] has introduced the class of $k$ - quasi- $\ast$ - class $A$ operators. An operator $T$ is said to be $k$ - quasi- $\ast$ - class $A$ operator on a complex Hilbert space $\mathcal{H}$ if $T^k(|T^2| - |T^*|^2)T^k \geq 0$ where $k$ is a natural number.

An operator $T$ is called normal if $T^*T = TT^*$ and $(p,k)$ - quasihyponormal if $T^{*k}((T^*T)^p - (TT^*)^p)T^k \geq 0$ $(0 < p \leq 1, k \in \mathbb{N})$. A. Aluthge [1], B.C. Gupta [6], S.C. Arora and P. Arora [3] introduced $p$ - hyponormal, $p$ - quasihyponormal and $k$ - quasihyponormal operators, respectively.

$p$ - hyponormal $\subset p$ - posinormal $\subset (p,k)$ - quasiposinormal,

$p$ - hyponormal $\subset p$ - quasihyponormal $\subset$

$(p,k)$ - quasihyponormal $\subset (p,k)$ - quasiposinormal

and

hyponormal $\subset k$ - quasihyponormal $\subset (p,k)$ - quasihyponormal $\subset (p,k)$ - quasiposinormal

for a positive integer $k$ and a positive number $0 < p \leq 1$.

If $T \in B(\mathcal{H})$, we shall write $N(T)$ and $R(T)$ for the null space and the range of $T$, respectively. Also, let $\sigma(T)$ and $\sigma_a(T)$ denote the spectrum and
the approximate point spectrum of $T$, respectively. An operator $T$ is called Fredholm if $R(T)$ is closed, $\alpha(T) = \dim N(T) < \infty$ and $\beta(T) = \dim \mathcal{H}/R(T) < \infty$. Moreover if $i(T) = \alpha(T) - \beta(T) = 0$, then $T$ is called Weyl. The essential spectrum $\sigma_e(T)$ and the Weyl $\sigma_W(T)$ are defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}$$

and

$$\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\},$$

respectively. It is known that $\sigma_e(T) \subset \sigma_W(T) \subset \sigma_e(T) \cup \text{acc } \sigma(T)$ where we write acc $K$ for the set of all accumulation points of $K \subset \mathbb{C}$. If we write iso $K = K \setminus \text{acc } K$, then we let

$$\pi_{00}(T) = \{\lambda \in \text{iso } \sigma(T) : 0 < \alpha(T - \lambda) < \infty\}.$$

We say that Weyl’s theorem holds for $T$ if

$$\sigma(T) \setminus \sigma_W(T) = \pi_{00}(T).$$

Let $\sigma_p(T)$ denotes the point spectrum of $T$, i.e., the set of its eigenvalues. Let $\sigma_{jp}(T)$ denotes the joint point spectrum of $T$. We note that $\lambda \in \sigma_{jp}(T)$ if and only if there exists a non-zero vector $x$ such that $Tx = \lambda x$, $T^*x = \overline{\lambda}x$. It is evident that $\sigma_{jp}(T) \subset \sigma_p(T)$. It is well known that, if $T$ is normal, then $\sigma_{jp}(T) = \sigma_p(T)$. If $T = U|T|$ is the polar decomposition of $T$ and $\lambda = |\lambda|e^{i\theta}$ be the complex number, $|\lambda| > 0$, $|e^{i\theta}| = 1$. Then $\lambda \in \sigma_{jp}(T)$ if and only if there exist a non-zero vector $x$ such that $Ux = e^{i\theta}$, $|T|x = |\lambda|x$. Let $\sigma_{ap}(T)$ denotes the approximate point spectrum of $T$, i.e., the set of all complex numbers $\lambda$ which satisfy the following condition: there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that $\lim_{n\to\infty}||(T - \lambda)x_n|| = 0$. It is evident that $\sigma_p(T) \subset \sigma_{ap}(T)$. Let $\sigma_{jap}(T)$ be the joint approximate point spectrum of $T$, then $\lambda \in \sigma_{jap}(T)$ if and only if there exists a sequence $\{x_n\}$ of unit vectors such that $\lim_{n\to\infty}||(T - \lambda)x_n|| = \lim_{n\to\infty}||(T^* - \overline{\lambda})x_n|| = 0$. It is evident that $\sigma_{jap}(T) \subset \sigma_{ap}(T)$ for all $T \in B(\mathcal{H})$. It is well known that, for a normal operator $T$, $\sigma_{jap}(T) = \sigma_{ap}(T) = \sigma(T)$.

An operator $T \in B(\mathcal{H})$ is said to have the single-valued extension property (or SVEP) if for every open subset $G$ of $\mathbb{C}$ and any analytic function $f : G \to \mathcal{H}$ such that $(T-z)f(z) \equiv 0$ on $G$, we have $f(z) \equiv 0$ on $G$. An operator $T \in B(\mathcal{H})$ is said to have Bishop’s property $(\beta)$ if for every open subset $G$ of $\mathbb{C}$ and every sequence $f_n : G \to \mathcal{H}$ of $\mathcal{H}$ - valued analytic functions such that $(T - z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of $G$, $f_n(z)$ converges
uniformly to 0 in norm on compact subsets of $G$. An operator $T \in B(H)$ is said to have Dunford’s property (C) if $H_T(F)$ is closed for each closed subset $F$ of $\mathbb{C}$. It is well known that

Bishop’s property ($\beta$) ⇒ Dunford’s property (C) ⇒ SVEP.

Let $T \in B(H)$ and let $\lambda_0$ be an isolated point of $\sigma(T)$. Then there exists a positive number $r > 0$ such that $\{\lambda \in \mathbb{C} : |\lambda - \lambda_0| \leq r\} \cap \sigma(T) = \{\lambda_0\}$. Let $\gamma$ be the boundary of $\{\lambda \in \mathbb{C} : |\lambda - \lambda_0| \leq r\}$. Then

$$E = \frac{1}{2\pi i} \int_\gamma (\lambda - T)^{-1} d\lambda,$$

is called the Riesz idempotent of $T$ for $\lambda_0$. Then it is well known that

$E^2 = E, \quad ET = TE, \quad \sigma(T|_{\text{ran}E}) = \{\lambda_0\} \quad \text{and} \quad \ker(T - \lambda_0 I) \subseteq \text{ran}E.$

In general, it is well known that the Riesz idempotent $E$ is not an orthogonal projection and a necessary and sufficient condition for $E$ to be orthogonal is that $E$ is self-adjoint. In [15], Stampfli showed that if $T$ satisfies the growth condition $G_1$, then $E$ is self-adjoint and $E(H) = \ker(T - \lambda_0)$. Recently, Jeon and Kim [11] and Uchiyama [18] obtained Stampfli’s result for quasi-class $A$ operators and paranormal operators. In general even if $T$ is a paranormal operator, the Riesz idempotent $E$ of $T$ with respect to $\lambda_0$ is not necessarily self-adjoint. We show that if $E$ is the Riesz idempotent for a nonzero isolated point $\lambda_0$ of the spectrum of a quasi-∗-class $A$ operator $T$, then $E$ is self-adjoint and $EH = \ker(T - \lambda_0) = \ker(T^* - \overline{\lambda_0}).$

2. Main Results

Jun Li and et al [12] have introduced quasi-∗-class $A$ operators and have proved many interesting properties of it.

**Lemma 2.1.** ([12, Theorem 2.2, Theorem 2.3]) (1) Let $T \in B(H)$ be quasi-∗-class $A$ operator and $T$ does not have a dense range, then

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$$

where $A = T|_{\text{ran}T}$ is the restriction of $T$ to $\text{ran}T$ and $A \in \ast\ast$-class $A$ operator. Moreover $\sigma(T) = \sigma(T_1) \cup \{0\}$.

(2) If $T$ is an quasi-∗-class $A$ operator and $M$ is its invariant subspace, then the restriction $T|_M$ of $T$ to $M$ is also an quasi-∗-class $A$ operator.
Lemma 2.2. [12, Theorem 2.4] Let $T \in B(H)$ is an quasi - * - class $A$ operator. If $\lambda \neq 0$ and $(T - \lambda)x = 0$ for some $x \in H$, then $(T - \lambda)^*x = 0$.

Lemma 2.3. Let $T \in B(H)$ is an quasi - * - class $A$ operator. Then $T$ is isoloid.

Proof. Let $T \in B(H)$ is an quasi - * - class $A$ operator with representation given in Lemma 2.1. Let $z$ be an isolated point in $\sigma(T)$. Since $\sigma(T) = \sigma(T_1) \cup \{0\}$, $z$ is an isolated point in $\sigma(T_1)$ or $z = 0$. If $z$ isolated point in $\sigma(T_1)$, then $z \in \sigma_p(T_1)$. Assume that $z = 0$ and $z \notin \sigma(T_1)$. Then for $x \in \ker(T_3)$, $-T_1^{-1}T_2x \oplus x \in \ker T$. This completes the proof.

Theorem 2.4. Let $A \in B(H)$ is an quasi - * - class $A$ operator and let $\lambda$ be a non-zero isolated point of $\sigma(A)$. Let $D_\lambda$ denote the closed disk that centered at $\lambda$ such that $D_\lambda \cap \sigma(A) = \{\lambda\}$. Then the Riesz idempotent

$$E = \frac{1}{2\pi i} \int_{\partial D_\lambda} (\lambda - A)^{-1}d\lambda$$

satisfies

$$EH = \ker(A - \lambda) = \ker((T - \lambda)^*).$$

In particular, $E$ is self adjoint.

Proof. If $A$ is quasi - * - class $A$ operator, then $\lambda$ is an eigenvalue of $A$ and $EH = \ker(A - \lambda)$ by Lemma 2.3. Since $\ker(A - \lambda) \subset \ker(A - \lambda)^*$ by Lemma 2.2, it suffices to show that $\ker(A - \lambda)^* \subset \ker(A - \lambda)$ . Since $\ker(A - \lambda)$ is a reducing subspace of $A$ by Lemma 2.2 and the restriction of a quasi - * - class $A$ operator to its reducing subspaces is also a quasi - * - class $A$ operator by Lemma 2.1, hence $A$ can be written as follows: $A = \lambda \oplus A_1$ on $H = \ker(A - \lambda) \oplus (\ker(A - \lambda))^\perp$, where $A_1$ is *-class $A$ with $\ker(A_1 - \lambda) = \{0\}$. Since

$$\lambda \in \sigma(A) = \{\lambda\} \cup \sigma(A_1)$$

is isolated, the only two cases occur, one is $\lambda \notin \sigma(A_1)$ and the other is that $\lambda$ is an isolated point of $\sigma(A_1)$ and this contradicts the fact that $\ker(A_1 - \lambda) = \{0\}$. Since $A_1$ is invertible as an operator on $(\ker(A - \lambda))^\perp$, $\ker(A - \lambda) = \ker(A - \lambda)^*$.

Next, we show that $E$ is self-adjoint. Since

$$EH = \ker(A - \lambda) = \ker(A - \lambda)^*,$$
we have \((z - A)^{-1}E = (z - \lambda)^{-1}E\). Therefore

\[
E^*E = -\frac{1}{2\pi i} \int_{\partial D} ((z - A)^{-1}E \, dz
= -\frac{1}{2\pi i} \int_{\partial D} (z - A)^{-1}E \, dz
= \left(\frac{1}{2\pi i} \int_{\partial D} (z - A)^{-1} \, dz\right) E
= E.
\]

This completes the proof. \(\square\)

3. Tensor product of quasi - * - class A operators

The tensor products \(T \otimes S\) preserves many properties of \(T, S \in B(\mathcal{H})\), but by no means all of them. Thus, whereas the normaloid property is invariant under tensor products; again, whereas \(T \otimes S\) is normal if and only if \(T\) and \(S\) are normal [10, 16], there exist paranormal operators \(T\) and \(S\) such that \(T \otimes S\) is not paranormal [4]. It is shown in [11] that \(T \otimes S\) is quasi-class A if and only if \(S, T\) are quasi-class A operators. In the following theorem we will prove a necessary and sufficient condition for \(T \otimes S\) to be quasi - * - class A operator where \(T\) and \(S\) are both non-zero operators.

Recall that \((T \otimes S)^*(T \otimes S) = T^*T \otimes S^*S\) and so, by the uniqueness of positive square roots, \(|T \otimes S|^r = |T|^r \otimes |S|^r\) for any positive rational number \(r\). From the density of the rationales in the real, we obtain \(|T \otimes S|^p = |T|^p \otimes |S|^p\) for any positive real number \(p\). If \(T_1 \geq T_2\) and \(S_1 \geq S_2\), then \(T_1 \otimes S_1 \geq T_2 \otimes S_2\) (see, [17])

**Theorem 3.1.** Let \(S, T \in B(\mathcal{H})\) be non-zero operators. Then \(T \otimes S\) is quasi - * - class A operator if and only if one of the following holds:

a) \(S\) and \(T\) are quasi - * - class A operators.

b) \(S^2 = 0\) or \(T^2 = 0\).

**Proof.** Since \(T \otimes S\) is quasi - * - class A operator if and only if \((T \otimes S)^*(|(T \otimes S)|^2 - (|(T \otimes S)|^2))((T \otimes S) \geq 0\)
\[\iff T^*|T^2| - |T^*|^2 T \otimes S^*|S^2|S + T^*|T^*|^2 T \otimes S^* (|S^2| - |S^*|^2)S \geq 0.\]
Hence the sufficiency is clear. Conversely, assume that $T \otimes S$ is quasi - $*$ - class $A$ operator. Then for every $x, y \in \mathcal{H}$ we have

$$
\langle T^* (|T^2| - |T^*|^2)Tx, x \rangle \langle S^* |S^2|Sy, y \rangle + \langle T^* |T^*|^2Tx, x \rangle \langle S^* (|S^2| - |S^*|^2)Sy, y \rangle \geq 0
$$

(3.1)

It suffices to prove that if (a) does not hold, then (b) holds. Suppose that $T^2 \neq 0$ and $S^2 \neq 0$.

To the contrary, assume that $T$ is not a quasi - $*$ - class $A$ operator, then there exists $x_0 \in \mathcal{H}$ such that

$$
\langle T^* (|T^2| - |T^*|^2)Tx_0, x_0 \rangle = \alpha < 0 \quad \text{and} \quad \langle T^* |T^*|^2Tx_0, x_0 \rangle = \beta > 0.
$$

From (3.1) we have

$$
\alpha + \beta \langle S^* |S^2|Sy, y \rangle \geq \beta \langle S^* |S^*|^2Sy, y \rangle
$$

(3.2)

for all $y \in \mathcal{H}$.

Thus $S$ is quasi - $*$ - class $A$ operator since $\alpha + \beta \leq \beta$. Using the Hölder-McCarthy inequality we have

$$
\langle S^* |S^2|Sy, y \rangle = \langle (S^2S^2)^{1/2}Sy, Sy \rangle \leq ||Sy||^{2(1-1/2)} \langle S^2S^2Sy, Sy \rangle^{1/2} = ||Sy|| ||S^3y||
$$

and

$$
\langle S^* |S^*|^2Sy, y \rangle = \langle SS^*Sy, Sy \rangle = \langle (S^*S)y, S^*Sy \rangle
$$

$$
= ||S^*Sy||^2.
$$

Thus

$$
(\alpha + \beta)||Sy|| ||S^3y|| \geq \beta ||S^*Sy||^2.
$$

(3.3)

Since $S$ is a quasi - $*$ - class $A$ operator, Lemma 2.1 imply that

$$
S = \begin{pmatrix} S_1 & S_2 \\ 0 & 0 \end{pmatrix} \quad \text{on} \quad H = \text{ran}(S^k) \oplus \ker S^{*k}.
$$

Then $S_1$ is $*$ - class $A$, $S_3^k = 0$ and $\sigma(S) = \sigma(S_1) \cup \{0\}$. Therefore (3.3) implies

$$
(\alpha + \beta)||S_1\eta|| ||S_3^3 \eta|| \geq \beta ||S_1^*S_1\eta||^2
$$

for all $\eta \in \text{ran} S^k$. Since $S_1$ is $*$ - class $A$ and $*$ - class $A$ is normaloid. Thus taking supremum on both sides of the above inequality, we have

$$
(\alpha + \beta)||S_1||^4 \geq \beta ||S_1^*S_1||^2.
$$
Therefore $S_1 = 0$. Since

$$S^2 = \begin{pmatrix} 0 & S_2 \\ 0 & 0 \end{pmatrix}^2 = 0.$$ 

Hence $S^{k+1} = 0$. This contradicts the assumption $S^2 \neq 0$. Hence $T$ must be a quasi - * - class $A$ operator. A similar argument shows that $S$ is also quasi - * - class $A$ operator. This completes the proof.

**References**


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