Confidence Intervals for the Difference of Coefficients of Variation for Lognormal Distributions and Delta-Lognormal Distributions

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Abstract

This paper presents new confidence intervals for the difference of the coefficients of variation for the lognormal distributions and the delta-lognormal distribution using the generalized pivotal approach (GPA) developed by Weerahandi [16] compared to the closed form method of variance estimation (CFM) presented by Zou et al. [18, 19]. We examine the performances of these confidence intervals in terms of the coverage probabilities and average lengths by Monte Carlo simulation.

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Keywords: coefficient of variation, confidence interval, generalized confidence intervals, lognormal distribution

1 Introduction

To compare the dispersions of several variables, the coefficient of variation (CV), defined to be the ratio of the standard deviation to the mean, is widely reported in the way that it does not depend on the variable measurement units; see, e.g., [2]. Therefore, many researches were studied on the confidence limits estimation for the CV of normal population; see, e.g., [14, 17, 13]. Many situations, the populations of interest may be reasonably described by lognormal distribution, validated in several applications; in particular, for analyzing data in biological and health researches. Confidence intervals for the CV of the lognormal populations by Koopmans et al. [5], using two-stage sequential sampling for specific situations that contain the prior and Verril [15], based on a
central chi-squared distribution. The idea to remove the skewness and to bring about symmetry distributed is to use the log-transformation. However in many situations, variables may be extremely sensitive to take a log-transformation because variables contain both terms of possible zero observations and positive observations which the positive term has a lognormal distribution. The distribution of population that mixes between lognormal data and zero values is called “a delta-lognormal distribution”. Numerous methods are available for constructing confidence intervals for the mean and the ratio or difference of two means for the delta-lognormal populations; see, e.g., [9, 11, 3, 7] but there have been no research done in a confidence intervals for the CV of a delta-lognormal distribution.

The purpose of this paper is to explore methods for constructing the confidence interval for the difference of CVs of the lognormal distributions and the difference of CVs for delta-lognormal distributions. It appears to be the first attempt to derive in these topics. We propose two methods to construct confidence intervals for the difference of CVs; the generalized pivotal approach, which was presented by Weerahandi [16], and the closed form method of variance estimation (CFM), presented by Zou et al. [18, 19].

2 The difference of coefficients of variation for lognormal populations

As according to statistical inference based on the normal distribution is well known, an established way to deal with the lognormal data is usually applied by a transformation that makes them normally distributed and uses the advantages of the normality in statistic inference; see, e.g., Niwitpong et al. [8] used the log-transformed data to the non-adaptive prediction interval, then transform the interval back. However, the way to directly deal with the lognormal data is more accurate. Let $X = (X_1, X_2, \ldots, X_n)$ be a random variable having a lognormal distribution, and $\mu$ and $\sigma^2$, respectively, are denoted by the mean and the variance of $Y$ where $Y = \ln(X) \sim N(\mu, \sigma^2)$. The probability density function of the lognormal distribution, $LN(\mu, \sigma^2)$, is

$$f(x, \mu, \sigma^2) = \begin{cases} \frac{1}{x\sigma\sqrt{2\pi}} \exp \left( -\frac{(\ln(x) - \mu)^2}{2\sigma^2} \right) & \text{; for } x > 0 \\ 0 & \text{; for } x \leq 0. \end{cases}$$ (1)

The CV for the lognormal population is modified by $CV = \sqrt{\text{Var}(X)}/E(X)$,

$$CV = \eta = \frac{\exp(2\mu + \sigma^2)(\exp(\sigma^2) - 1)}{\exp(\mu + \sigma^2/2)}$$
Confidence intervals for the difference of coefficients of variation

\[ = \sqrt{\exp (\sigma^2) - 1} \] 

where \( Var(X) = \exp (2\mu + \sigma^2)(\exp (\sigma^2) - 1) \) denoted by the population variance of \( X \) and \( E(X) = \exp (\mu + \sigma^2/2) \) denoted by the population expectation of \( X \). Consequently, the CV depends only on the parameter \( \sigma^2 \).

Suppose that there are two independent populations \( (i = 1, 2) \) of interest \( X_i = (X_{i1}, X_{i2}, ..., X_{im_i}) \) which \( Y_i = \ln(X_i) \sim N(\mu_i, \sigma_i^2) \). Equivalently, assume \( X_i \) have lognormal distributions, \( LN(\mu_i, \sigma_i^2) \), which the CV of each group is \( \eta_i = \sqrt{\exp (\sigma_i^2) - 1} \). The difference of two CVs, \( \eta_1 - \eta_2 \), is:

\[
\psi = \eta_1 - \eta_2 = \left( \sqrt{\exp (\sigma_1^2)} - 1 \right) - \left( \sqrt{\exp (\sigma_2^2)} - 1 \right).
\] 

The confidence intervals for the parameter \( \psi \) can be constructed by the following methods.

2.1 The generalized pivotal approach, GPA

We introduce here our generalized pivotal approach for the difference of CVs for lognormal populations. As Weerahandi [16] defined a generalized pivotal (GP) as a statistic that has a distribution free of unknown parameters and does not depend on nuisance parameters. The idea to construct the confidence intervals for \( \psi \) is used through the potential generalized pivotal quantity based directly on observed values of \( X_i \). Suppose \( S_i^2 \) are denoted by the sample variance for log-transformed data \( Y_i = \ln(X_i) \), for \( i = 1, 2 \) and \( s_i^2 \) are denoted by the observed sample variance from the population \( i^\text{th} \). The generalized pivotal quantities for \( \sigma_i^2 \) are

\[
R_{\sigma_i^2} = \frac{(n_i - 1)s_i^2}{U_i}
\]

where \( U_i \) are the chi-square distribution with \( n_i - 1 \) degree of freedom,

\[
U_i = \frac{(n_i - 1)S_i^2}{\sigma_i^2}.
\]

Obviously, both generalized pivotal quantities \( R_{\sigma_1^2} \) and \( R_{\sigma_2^2} \) are independent. The generalized pivotal quantity for \( \psi \) is given by

\[
R_{\psi} = \left( \sqrt{\exp (R_{\sigma_1^2})} - 1 \right) - \left( \sqrt{\exp (R_{\sigma_2^2})} - 1 \right).
\]

Note that for given \( s_i^2, (i = 1, 2) \), the following holds:

(i) the distribution of \( R_{\psi} \) is independent of any unknown parameter; and

(ii) the observed pivotal does not depend on the nuisance parameters and the value \( R_{\psi} \) is equal to \( \psi \) as \( s_i^2 = S_i^2, (i = 1, 2) \). Therefore \( R_{\psi} \) is a generalized...
pivotal quantity for constructing confidence interval for $\psi$ and its quantiles may be used to construct on the basis of $R_\psi$. If $R_\psi(1 - \alpha)$ is denoted by the $100(1 - \alpha)th$ percentile of the distribution of $R_\psi$, then $R_\psi(1 - \alpha)$ is the $100(1 - \alpha)\%$ generalized upper confidence interval for $\psi$. Thus,

$$CI_1 = [\psi_L, \psi_U] = [R_\psi(\alpha/2), R_\psi(1 - \alpha/2)]$$ (4)

is a $100(1 - \alpha)\%$ two-sided GCI for the difference of coefficients of variation. Thus the coverage probability of the GCI can be computed using the Algorithm 1.

**Algorithm 1.** For given $n_i, \mu_i, \sigma_i^2, (i = 1, 2)$, and $0 < \alpha < 1$, the GCI can be computed by the following steps.

1. Generate set $x_{i1}, x_{i2}, \ldots, x_{in_i}$ from $LN(\mu_i, \sigma_i^2)$, set $y_{ij} = \ln(x_{ij}), i = 1, 2$, and $j = 1, 2, \ldots, n_i$.
2. Compute $(\bar{y}_1, s_1^2)$ and $(\bar{y}_2, s_2^2)$.
3. Generate $U_1 \sim \chi^2_{(n_1 - 1)}$ and $U_2 \sim \chi^2_{(n_2 - 1)}$.
4. Compute both generalized pivotal quantities $R_{\sigma_1^2}$ and $R_{\sigma_2^2}$.
5. Compute $R_\psi$ following (4).
6. Repeat Step 3-5 a total of $m_1$ times and obtain an array of $R_\psi$’s.
7. Rank this array of $R_\psi$’s from small to large.
8. If the $100(1-\alpha/2)th$ percentile of $R_\psi$’s is greater than $\psi$ or the $100(\alpha/2)th$ percentile $R_\psi$’s is smaller than $\psi$, set $K_j = 1$ else set $K_j = 0$, $j = 1, 2, \ldots, m_2$.
9. Repeat Step 1-8 a total of $m_2$ times and obtain an array of $K_j$’s.

The value $\frac{1}{m_2} \sum_{j=1}^{m_2} K_j$ is an estimate of a coverage probability of a $100(1 - \alpha)\%$ two-sided GCI for the difference of CVs, $\psi$.

### 2.2 The closed form method of variance estimation, CFM

Recently, the new method called the closed form method of variance estimation (CFM) was presented by Zou and his colleagues in many papers; see, e.g., [18, 19, 4]. Their strategy is to recover variance estimates from confidence interval and then approximate confidence intervals for functions of parameters.
Using the central limit theorem, a general approach to set two-sided 100(1 − α)% confidence interval for θ₁ − θ₂ is given by

\[
(\hat{\theta}_1 - \hat{\theta}_2) \pm Z_{(1-\alpha/2)} \sqrt{\text{Var}(\hat{\theta}_1) + \text{Var}(\hat{\theta}_2)}
\]

where \( Z_{(1-\alpha/2)} \) is the \((1 - \alpha/2)\) quantile of the standard normal distribution, \( \hat{\theta}_i \) are estimators of \( \theta_i \), and \( \text{Var}(\hat{\theta}_i) \) are statistically independent of each other, where \( i = 1, 2 \). Since \( \text{Var}(\hat{\theta}_i) \) is unknown, the method to improve of estimating variance of \( (\hat{\theta}_1 - \hat{\theta}_2) \) in the neighborhood of the limits \( (L, U) \) for \( \theta_1 - \theta_2 \) is estimated by setting of \( \min(\theta_1 - \theta_2) \) for \( L \) and that of \( \max(\theta_1 - \theta_2) \) for \( U \). Suppose that 100(1 − α)% two-sided confidence intervals for \( \theta_i \) are given by \((l_i, u_i)\). The plausible values of \( \theta_1 - \theta_2 \) are possible among the plausible minimum \((l_1 + l_2)\) and the plausible maximum \((u_1 + u_2)\).

Zou et al. [18, 19, 4] proposed the confidence interval \((L', U')\) for \( \theta_1 - \theta_2 \) where

\[
L' = (\hat{\theta}_1 - \hat{\theta}_2) - \sqrt{(\hat{\theta}_1 - l_1)^2 + (u_2 - \hat{\theta}_2)^2}
\]

and

\[
U' = (\hat{\theta}_1 - \hat{\theta}_2) + \sqrt{(u_1 - \hat{\theta}_1)^2 + (\hat{\theta}_2 - l_2)^2}.
\]

Since the difference of two CVs is \( \psi = \sqrt{\exp(\sigma_1^2) - 1} - \sqrt{\exp(\sigma_2^2) - 1} \), this function can be given by two unknown parameters \( \theta_1, \theta_2 \) where \( \theta_1 = \sqrt{\exp(\sigma_1^2) - 1} \) and \( \theta_2 = \sqrt{\exp(\sigma_2^2) - 1} \). Then the estimators of \( \theta_1 \) and \( \theta_2 \) are \( \hat{\psi} = \sqrt{\exp(S_1^2) - 1} - \sqrt{\exp(S_2^2) - 1} \), respectively, and the estimator of \( \psi \) is \( \hat{\psi} = \sqrt{\exp(S_1^2) - 1} - \sqrt{\exp(S_2^2) - 1} \).

It is easy to see that a 100(1 − α)% two-sided confidence intervals for \( \theta_1 \) and \( \theta_2 \) based on the exact approach are

\[
CI_{\theta_1} = [\theta_{1l}, \theta_{1u}] = \left[ \sqrt{\exp\left(\frac{(n_1 - 1)S_1^2}{\chi^2_{(n_1 - 1), (1-\alpha/2)}}\right)} - 1, \sqrt{\exp\left(\frac{(n_1 - 1)S_1^2}{\chi^2_{(n_1 - 1), (\alpha/2)}}\right)} - 1 \right]
\]

and

\[
CI_{\theta_2} = [\theta_{2l}, \theta_{2u}] = \left[ \sqrt{\exp\left(\frac{(n_2 - 1)S_2^2}{\chi^2_{(n_2 - 1), (1-\alpha/2)}}\right)} - 1, \sqrt{\exp\left(\frac{(n_2 - 1)S_2^2}{\chi^2_{(n_2 - 1), (\alpha/2)}}\right)} - 1 \right]
\]

Using the idea of the CFM to construct the confidence interval for the difference of CVs, \( \psi \), a 100(1 − α)% two-sided confidence interval for \( \psi \) is expressed the following equation,
\[ CI_2 = \left[ \hat{\psi} - \sqrt{(\theta_1 - \hat{\theta}_1)^2 + (\theta_2 - \hat{\theta}_2)^2}, \hat{\psi} + \sqrt{(\theta_1 - \hat{\theta}_1)^2 + (\theta_2 - \hat{\theta}_2)^2} \right]. \quad (5) \]

3 The difference of coefficients of variation for delta-lognormal populations

Populations containing many zeros also appear in medical applications [12] that can take any nonnegative values. Additionally, Kvanli et al. [6] gave examples of such situations include the following areas.

1) Reliability: A sample of 100 units produces 40 that fail on installation (a zero lifetime is recorded) and 60 that fail after installation.

2) Insurance: A sample of 150 policyholders is obtained, and the total dollar amount of claims filed during the past year is recorded. The sample contains 32 policyholders who filed a claim (the claim amount is recorded) and 118 individuals who did not file a claim during this time (a zero is recorded).

3) Meteorology: A sample of rainfall data from a given geographical region produces 95 days of no precipitation and 25 days with rainfall amounts that appear to follow a continuous distribution.

Let \( T = (T_1, T_2, \ldots, T_n) \) be a positive random variable having a lognormal distribution \( \text{LN}(\mu, \sigma^2) \), and \( Y = \ln(T) \sim N(\mu, \sigma^2) \) where \( \mu \) and \( \sigma^2 \) denote the mean and variance of \( Y \), respectively. The p.d.f. of \( T \) is

\[
f(t, \mu, \sigma^2) = \begin{cases} 
\frac{1}{t\sigma\sqrt{2\pi}} \exp \left( -\frac{(\ln(t) - \mu)^2}{2\sigma^2} \right) & ; \text{if } t > 0 \\
0 & ; \text{if otherwise.}
\end{cases}
\]

Suppose \( X = (X_1, X_2, \ldots, X_n) \) be a non-negative random sample from a delta-lognormal population, \( \Delta(\mu, \sigma^2, \delta) \), where the number of zero observations have the binomial distribution, \( n(0) \sim B(n, \delta) \). Define \( \delta = P(X = 0) \),

\[
\hat{\delta} = \frac{n(0)}{n}, \quad \hat{\mu} = \frac{1}{n(1)} \sum_{j=1}^{n(1)} x_i, \quad \hat{\sigma}^2 = \frac{1}{n(1) - 1} \sum_{j=1}^{n(1)} (x_i - \hat{\mu})^2 \quad \text{and} \quad n(0) + n(1) = n
\]

where \( n(0) \) and \( n(1) \) are the number of zero observed values and positive observed values of the random variables, respectively. Thus the distribution function of the delta-lognormal population, \( \Delta(\mu, \sigma^2, \delta) \), is shown by Tian and Wu [12] as follows:

\[
G(x, \mu, \sigma^2, \delta) = \begin{cases} 
\delta & ; \text{if } x = 0 \\
\delta + (1 - \delta)F(x, \mu, \sigma^2) & ; \text{if } x > 0,
\end{cases}
\]

(7)
Confidence intervals for the difference of coefficients of variation

where \( F(x, \mu, \sigma^2) \) is the lognormal cumulative distribution function.

Suppose we have two delta-lognormal populations of interest. Let \( X_1 = (X_{11}, X_{12}, \ldots, X_{1n_1}) \) denote a random sample from the first population having a delta-lognormal \((\mu_1, \sigma_1^2, \delta_1)\) distribution and let \( X_2 = (X_{21}, X_{22}, \ldots, X_{2n_2}) \) denote a random sample from the second population having a delta-lognormal distribution, \( \Delta(\mu_2, \sigma_2^2, \delta_2) \). Assume that \( X_{kj} \) take on the a value of zero with probability \( \delta_k \) where \( k = 1, 2 \) and \( j = 1, \ldots, n_k \).

The coefficient of variation of \( X_k \) are

\[
CV_{X_k} = \lambda_k = \sqrt{\frac{\exp(\sigma_k^2) + \delta_k - 1}{1 - \delta_k}},
\]

where \( E(X_k) = (1 - \delta_k) \exp(\mu_k + \sigma_k^2/2) \), and

\[
Var(X_k) = (1 - \delta_k) \exp(2\mu_k + \sigma_k^2)[\exp(\sigma_k^2) + \delta_k - 1].
\]

The difference of the coefficient of variation, \( \eta_1 - \eta_2 \), is simply:

\[
\psi_1 = \lambda_1 - \lambda_2 = \sqrt{\frac{\exp(\sigma_1^2) + \delta_1 - 1}{1 - \delta_1}} - \sqrt{\frac{\exp(\sigma_2^2) + \delta_2 - 1}{1 - \delta_2}}.
\]

We can construct confidence intervals for \( \psi \) by two methods obtaining as follows.

### 3.1 The Generalized Pivotal Approach, GPA

Suppose \( S_k^2 \) denote the sample variance for log-transformed data of non-zeros \( Y_{kj} = \ln(X_{kj}) \) and \( s_k^2 \) denote the observed sample variance. We have the generalized pivotal

\[
R_{\sigma_k^2} = \frac{(n_k(1) - 1)s_k^2}{U_k}
\]

where \( U_k \) is the chi-square distribution with \( n_k(1) - 1 \) degree of freedom,

\[
U_k = \frac{(n_k(1) - 1)S_k^2}{\sigma_k^2}.
\]

Obviously, both generalized pivotal quantities \( R_{\sigma_1^2} \) and \( R_{\sigma_2^2} \) are independent.

As according to the Agresti-Coull normal approximation [1] to the binomial distribution in applying to the generalized pivotal approach; see, e.g., Chen and Zhou [3], we have

\[
\hat{\delta}_k = \frac{n_k(0) + \frac{1}{2}z_\alpha^2}{n_k + z_\alpha^2}
\]

and define \( f_{\delta_k} \) as

\[
f_{\delta_k}(n_k, n_k(0)) = \hat{\delta}_k - Z \sqrt{\hat{\delta}_k(1 - \hat{\delta}_k)/(n_k + z_\alpha^2)}.
\]
Additionally, define $W_k$ as

$$W_k = \begin{cases} 
0 & \text{if } f_{\delta_k}(n_k, n_k(0)) < 0, \\
\text{Uniform}(99/100, 1) & \text{if } f_{\delta}(n_k, n_k(0)) > 1, \\
f_{\delta_k}(n_k, n_k(0)) & \text{otherwise}.
\end{cases}$$

The generalized pivotal quantity for $\psi$ is given by

$$R_{\psi_1} = \sqrt{\frac{[\exp (R_{\sigma_1^2}) + W_1 - 1]}{(1 - W_1)}} - \sqrt{\frac{[\exp (R_{\sigma_2^2}) + W_2 - 1]}{(1 - W_2)}}.$$ 

Note that for given $s_k^2$, (k=1, 2), the following holds:

(i) the distribution of $R_{\psi_1}$ is independent of any unknown parameters; and
(ii) the value of $R_{\psi_1}$ is equal to $\psi_1$ as $S_k^2 = s_k^2$, (k=1, 2). Therefore $R_{\psi_1}$ is a generalized pivotal quantity for constructing the confidence interval of $\psi_1$ and its quantiles may be used to construct confidence limits for $\psi_1$.

Hence the GCI for $\psi_1$ can be constructed on the basis of $R_{\psi_1}$, if $R_{\psi_1}(1 - \alpha)$ denotes the 100$(1 - \alpha)$th percentile of the distribution of $R_{\psi_1}$, then $R_{\psi_1}(1 - \alpha)$ is the 100$(1 - \alpha)$% generalized upper confidence interval for $\psi_1$. Thus

$$CI_3 = [\psi_L, \psi_U] = [R_{\psi_1}(\alpha/2), R_{\psi_1}(1 - (\alpha/2))].$$ (10)

is a 100$(1 - \alpha)$% two-sided GCI for the CV of a delta-lognormal distribution, $\Delta(\mu, \sigma^2, \delta)$. It is necessary to simulate the coverage probability to study the behavior of the confidence interval. Such simulation results are reported in Section 4.

### 3.2 The closed form method of variance estimation, CFM

Since the difference of two CVs is

$$\psi_1 = \sqrt{\frac{[\exp (\sigma_1^2) + \delta_1 - 1]}{(1 - \delta_1)}} - \sqrt{\frac{[\exp (\sigma_2^2) + \delta_2 - 1]}{(1 - \delta_2)}}.$$ 

Let, as in section 2.2, $\theta_1 = \sqrt{\frac{[\exp (\sigma_1^2) + \delta_1 - 1]}{(1 - \delta_1)}}$ and $\theta_2 = \sqrt{\frac{[\exp (\sigma_2^2) + \delta_2 - 1]}{(1 - \delta_2)}}$ then the estimators of $\theta_1$ and $\theta_2$ are

$$\hat{\theta}_1 = \sqrt{\frac{[\exp (\hat{\sigma}_1^2) + \hat{\delta}_1 - 1]}{(1 - \hat{\delta}_1)}}$$ and $\hat{\theta}_2 = \sqrt{\frac{[\exp (\hat{\sigma}_2^2) + \hat{\delta}_2 - 1]}{(1 - \hat{\delta}_2)}}$, respectively.

Using the idea of the CFM to construct the confidence interval for the difference of CVs for delta-lognormal distribution, $\psi_1$, a 100$(1 - \alpha)$% two-sided confidence interval for $\psi_1$ is expressed the following equation,

$$CI_{\psi_1} = [\psi_{1L}, \psi_{1U}]$$ (11)
where
\[
\psi_{1l} = (\hat{\theta}_1 - \hat{\theta}_2) - \sqrt{\left(\hat{\theta}_1 - l_1\right)^2 + \left(u_2 - \hat{\theta}_2\right)^2},
\]
\[
\psi_{1u} = (\hat{\theta}_1 - \hat{\theta}_2) + \sqrt{\left(u_1 - \hat{\theta}_1\right)^2 + \left(\hat{\theta}_2 - l_2\right)^2},
\]

and
\[
l_1 = \left[\exp \left(\frac{(n_1(1) - 1)s_1^2}{\chi^2_{(n_1(1) - 1), (1 - \hat{\delta}_1)}}\right) + \hat{\delta}_1 - 1\right] \frac{1}{(1 - \hat{\delta}_1)},
\]
\[
l_2 = \left[\exp \left(\frac{(n_2(1) - 1)s_2^2}{\chi^2_{(n_2(1) - 1), (1 - \hat{\delta}_2)}}\right) + \hat{\delta}_2 - 1\right] \frac{1}{(1 - \hat{\delta}_2)},
\]
\[
u_1 = \left[\exp \left(\frac{(n_1(1) - 1)s_1^2}{\chi^2_{(n_1(1) - 1), (\frac{\hat{\alpha}_2}{2})}}\right) + \hat{\delta}_1 - 1\right] \frac{1}{(1 - \hat{\delta}_1)},
\]
\[
u_2 = \left[\exp \left(\frac{(n_2(1) - 1)s_2^2}{\chi^2_{(n_2(1) - 1), (\frac{\hat{\alpha}_2}{2})}}\right) + \hat{\delta}_2 - 1\right] \frac{1}{(1 - \hat{\delta}_2)}.
\]

Notice that \(\hat{\delta}_k = \frac{n_{k(0)}}{n_k}\) the number of zero observations have a binomial distribution \((n_k, \delta_k)\).

4 Simulation Studies

The simulation studies are carried out to evaluate coverage probabilities and average lengths of each confidence interval. The nominal value of 0.95 is calculated based on 10,000 replications and for the generalized pivotal computations 5,000 pivotal quantities are used. All computer simulations, in variety of parameters, are studied by using written functions in \textit{R} statistical programming environment [10].

4.1 The difference of CVs for lognormal populations

In this part, the simulations of confidence intervals for the difference of CVs, \(\psi\), for lognormal distributions are compared in a variety of \(\eta_1\) and \(\eta_2\). We consider \((n_1, n_2) = \{(100, 100), (500, 500), (1, 000, 1000), (100, 500), (500, 1, 000)\}\), \((\eta_1, \eta_2) = \{(0.1, 0.1), (0.33, 0.33), (0.67, 0.67), (0.1, 0.33), (0.33, 0.67)\}\) and two different sets of \((\mu_1, \mu_2) = \{(0, 0), (0, 10)\}\). The performances of the GCI
method and the CFM are evaluated by using coverage probabilities and average lengths for all combination parameters and are presented at 0.95 nominal value in Table 1.

From Table 1, these results clearly show that the GPA generally provides coverage probabilities close to the nominal level than that of the CFM. The average length of the GCI is as short as that of the CFM. As these results, the GPA appears to be more reliably consistent than that of the CFM.

### 4.2 The difference of CVs for delta-lognormal populations

The simulations of confidence intervals for the difference of two CVs, $\psi$, for delta-lognormal distributions with $\mu_1 = 0$ and $\mu_2 = 0$ are compared in a variety of $n_1, n_2, \lambda_1, \lambda_2, \delta_1, \delta_2$. We consider $(n_1, n_2) = \{(100, 100), (200, 200), (500, 500), (1000, 1000)\}$; $(\eta_1, \eta_2, \delta_1, \delta_2) = \{(0.2, 0.2, 0.01, 0.01), (0.2, 0.2, 0.02, 0.02), (0.35, 0.35, 0.01, 0.01), (0.35, 0.35, 0.1, 0.1), (0.67, 0.67, 0.01, 0.01), (0.67, 0.67, 0.1, 0.1), (0.67, 0.67, 0.3, 0.3), (0.35, 0.67, 0.01, 0.01), (0.35, 0.67, 0.1, 0.1), (0.35, 0.67, 0.3, 0.3)\}$.

The performances of the GCI method and the CFM are evaluated by using coverage probabilities and average lengths for all combination parameters and are presented at 0.95 nominal value in Table 1.

From Table 2, the results indicate that the GCF and CFM methods generally provides coverage probability close to the nominal level of 0.95. The average length of the GCI is quite smaller than that of the CFM. These results show that the average lengths of both approaches tend to be clearly wider as larger CVs and tend to be slightly narrower as larger sample sizes. Moreover, the GCI appears to be easier and more preferable for estimation the difference of CVs for delta-lognormal distribution.

### 5 Discussion

In this paper, the confidence intervals for the difference of CVs for lognormal distributions and the delta-lognormal distributions are studied. This study proposed the new confidence intervals for the difference of CVs based on: a) the generalized pivotal approach (GPA) and b) the closed form method of variance estimation (CFM). The performances of these confidence intervals were assessed in the term of coverage probabilities and average lengths through simulation studies.

The simulation study for the difference of CVs for lognormal distributions indicates that the the performances of GPA is better than that of the CFM in term of coverage probability and average length. In the case of the difference of CVs for delta-lognormal distribution which consists a complex function of
parameters of parameters $\delta_k$, the average length of the GPA is smaller than that of that CFM and the coverage probability of the GPA appears to be more consistent than that of the CFM.

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**References**


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Table 1: Coverage probabilities and average lengths for $\psi$ at the 0.95 nominal value.

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$^a$ is the coverage probability of confidence interval.

$^b$ is the average length of confidence interval.

$^c$ is the ratio of the GCI’s average length to the CFM’s average length.
Table 2: Coverage probabilities and average lengths for $\psi$ at the 0.95 nominal value.

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<td>0.121</td>
<td>0.947</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.30</td>
<td>0.30</td>
<td>0.951</td>
<td>0.072</td>
<td>0.945</td>
</tr>
</tbody>
</table>

$^a$ is the coverage probability of confidence interval.

$^b$ is the average length of confidence interval.

$^c$ is the ratio of the GCI’s average length to the CFM’s average length.