Analysis of M/G/1 Feedback Queue with Three Stage and Multiple Server Vacation

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Abstract

We consider an M/G/1 queue with three stages of service with different general service time distributions. Bernoulli feedback and multiple server vacation, where the arrivals are Poisson. After first stage service, the server must provide the third stage service. However after the completion of third stage of service, if the customer is dissatisfied with his service, he can immediately join the tail of the queue as a feedback customer with probability \( p \). Otherwise the customer may depart forever from the system with probability \( q = 1 - p \). In addition the server takes vacation each time the system becomes empty and the vacation periods are assumed to be general. On returning from vacation if the server again founds no customer waiting in the system, then it again goes for vacation .The server continues to go for vacation until he finds at least one customer in the system. We obtain steady state probability generating functions for the number of the queue length for various states of the server. The average number of customers and the average waiting time in the queue as well the system are derived. Some special cases of interest are also discussed.
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1. Introduction

Queueing models with server vacations have been investigated by many authors due to their various applications in production, inventory system, communication systems, banking services, computer systems etc. We consider here a three stage $M/G/1$ Bernoulli feedback queue with multiple server vacation. When the system is empty, the server leaves the system and returns after a vacation of random duration. In multiple vacation policy, the server keeps on taking vacations after returning from vacation period, until there is at least one customer present in the system.

Queueing models with vacations have been investigated by many authors including Keilson and Servi [12], Cramer [21], Scholl and kleinrock [22], Shanthakumar [14], Doshi [4] and [5], Madan [16], [17], [18]. Choudhury and Madan [8] have studied a queueing system with Bernoulli schedule server vacation. Chae et al. [15], Chang and Takine [28] and Igaki [23] have studied queues with generalized vacations. Vacation queue with $c$ servers has been studied by Tian et al. [25]. Choudhury and Borthakur [7] and Hur and Ahn [27] have studied vacation queues with batch arrivals. Queue with multiple vacations has been studied by Tian and Zhang [24].

Recently there have been several contributions considering queueing systems of $M/G/1$ type in which the server may provide a second phase of service. One may refer to Bertsimas and Papaconstantinou [6], Madan [19], [20], Choudhury [9], [10] and [11], Medhi [13], Kalyanaraman [26], Krishna Kumar [2], Badamchi and Shahkar[1] have also studied a single server queue with two phase queueing system with Bernoulli feedback and Bernoulli schedule server vacation.

In this paper, we analyze a single server queue with three stages of service subject to Bernoulli feedback and multiple server vacation where the arrivals are Poisson. However after the completion of third stage of service, if the customer is dissatisfied with his service for certain reason or if he received unsuccessful service, then the customer may immediately join the tail of the original queue with probability $p$ $(0 \leq p < 1)$. Otherwise the customer may depart forever from the system with probability $q = (1-p)$. If there is no customer waiting in the system then the server goes for vacation with random duration. It follows general distribution. On returning from vacation, if the server again founds no customer
waiting in the system, then it goes for another vacation. The server continues to
go for vacation until he finds at least one customer in the system following
multiple server vacation.

The rest of the paper is organized as follows. In the next section, we
describe the mathematical description of our model. In Section 3, we give the
definitions and equations governing the system and obtain the explicit steady
state results for the probability generating functions of the queue size in Section
4. Expected number of customers in the queue as well in the system have been
found in Section 5. Mean waiting time has been found in Section 6. Finally some
special cases of interest have been derived in Section 7.

2. Mathematical Description of the Model

We assume the following to describe the queueing model of our study.

- Customers arrive at the system one by one in a Poisson stream with
  arrival rate \( \lambda > 0 \).
- Each customer undergoes three stages of service provided by a single
  server on a first come first served basis. The service times of the two
  stages follow different general (arbitrary) distributions with distribution
  function \( B_j(v) \) and the density function \( b_j(v) \).
- \( B_j(v) \) for \( j = 1, 2, 3 \) assuming that they have finite moments \( E(B_i^l) \)
  for \( l \geq 1 \) and \( i = 1, 2, 3 \).
- As soon as the third stage of a customer is completed and if the customer
  is dissatisfied
  With his service for certain reason or if he received unsuccessful
  service, the customer may immediately join the tail of the original queue
  with probability \( p \) (\( 0 \leq p < 1 \)) Otherwise the customer may depart
  forever from the system with probability \( q = (1 - p) \).
- Let \( \mu_i(x)dx \) be the conditional probability of completion of the \( i^{th} \)
  stage of service during the time interval \( (x, x+dx) \), given that the elapsed
time is \( x \), so that
  \[ \mu_i(x) = \frac{b_i}{1-B_i(x)}, i = 1, 2, 3 \ldots (2.1) \]

And therefore,

\[ b_i(v) = \mu_i(v)e^{-\int_0^\infty \mu_i(x)dx}, i = 1, 2, 3 \ldots (2.2) \]

- If there is no customer waiting in the system then the server goes for vacation
  with random duration. It has general distribution with distribution function
  \( V(x) \) and the probability density function \( v(x) \). Also let \( E(v^l) \) be the \( l^{th} \)
finite moment of \( V \) where \( l \geq 1 \). On returning from vacation, if the server again founds no
customer waiting in the system, then it goes for vacation again. The server
continues to go for vacation until he finds at least one customer in the system. So,
the server takes multiple vacation.

• Let \( Y(x)dx \) be the conditional probability of completion of the vacation during
the time interval \((x, x+dx)\), given that the elapsed vacation time is \( x \), so that
\[
Y(x) = \frac{v(x)}{1 - v(x)}
\]
and therefore,
\[
v(t) = Y(t)e^{-\int_0^t(Y(x)dx)}
\]

• The customer both newly arrived and those that are fed back are served in the
order in which they join the tail of the original queue. Also service time for a
feedback customer is independent of its previous service time.

• The customers are served according to the first come, first serve rule.

• Various stochastic processes involved in the system are independent of each
other.

3. Definitions and Equations Governing the System

We first derive the system state equations for the queue size distribution at
stationary point of time by treating three stages of service time as supplementary
variables. Then we solved these equations to derive the probability generating
function (PGF) for it.

Assuming that the system is in steady state condition and define \( N_q(t) \)
as the queue size (excluding one in service) at time \( t \), \( B_i^0(t) \) as the elapsed \( i^{th} \)
stage of service for at time \( t \) and \( i \)\( \epsilon \{1, 2, 3\} \) respectively. Also \( V_0(t) \) denote the
elapsed vacation time at \( t \). For \( i = 1, 2, 3 \), we introduce the random variable \( Y(t) \)
as follows

\[
Y(t) = \begin{cases} 
1, & \text{if the system is busy with first stage of service at time } t, \\
2, & \text{if the system is busy with second stage of service at time } t, \\
3, & \text{if the system is busy with third stage of service at time } t, \\
4, & \text{if the server is on vacation at time } t.
\end{cases}
\]

Then the supplementary variables \( B_i^0(t) \) for \( i \in \{1, 2\} \) and \( V_0(t) \) are introduced
in order to obtain a bivariate Markov process \( \{N_q(t), L(t)\} \) where

\[
L(t) = B_1^0(t), \text{ if } Y(t) = 1
\]
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\[ B_2^0(t), \text{ if } Y(t) = 2 \]
\[ B_3^0(t), \text{ if } Y(t) = 3 \]
\[ V_0^0(t), \text{ if } Y(t) = 4 \]

And define

\[ V_n(x) = \lim_{t \to \infty} P[N_q(t) = n, L(t) = V^0(t); x < V^0(t) \leq (x + dx)]; \]
\[ x > 0, n \geq 0, (3.3) \]

\[ P^{(i)}(x)dx = \lim_{t \to \infty} P[N_q(t)n, L(t) = B_i^0(t); x < B_i^0(t) \leq (x + dx)]; \]
\[ x > 0, n \geq 0, \text{ for } i \in \{1, 2, 3\}, (3.4) \]

Assume that

\[ V(0) = 0, V(\infty) = 1 \]

And for i=1,2,3

\[ B_i(0), B_i(\infty) = 1 \]

Also \( V(x) \) and \( B_i(x) \) are continuous at \( x=0 \). Then we have the hazard rate functions of \( V \) and \( B_i(i = 1, 2, 3, i \in \{1, 2, 3\}) \)s given in equations (2.3) and (2.1) respectively. So with the assumption that steady state exist, we have

\[ P^{(i)}(x)dx = \lim_{t \to \infty} P^{(i)}(x,t)dx, \quad i = 1, 2, 3, \quad x > 0, n \geq 0 \]

\[ V_n(x) = \lim_{t \to \infty} V_n(x,t)dx, \quad x > 0, n \geq 0 \]

And define:

\( P_n^{(1)}(x) \) = Steady state probability that there are \( n(\geq 0) \) customers in the queue excluding one in the first stage of service and the elapsed service time of this customer is \( x \). Accordingly, \( P_n^{(1)} = \int_0^\infty P_n^{(1)}(x)dx \) denotes the steady state probability that there are \( n \geq 0 \) customers in the queue excluding one in the first stage of service irrespective of the value of \( x \).

\( P_n^{(2)}(x) \) = Steady state probability that there are \( n(\geq 0) \) customers in the queue excluding one in the second stage of service and the elapsed service time of this customer is \( x \). Accordingly, \( P_n^{(2)} = \int_0^\infty P_n^{(2)}(x)dx \) denotes the steady state probability that there are \( n \geq 0 \) customers in the queue excluding one in the second stage of service irrespective of the value of \( x \).
\( P_n^{(3)}(x) = \) Steady state probability that there are \( n(\geq 0) \) customers in the queue excluding one in the third stage of service and the elapsed service time of this customer is \( x \). Accordingly, \( P_n^{(3)} = \int_0^\infty P_n^{(3)}(x)dx \) denotes the steady state probability that there are \( n \geq 0 \) customers in the queue excluding one in the third stage of service irrespective of the value of \( x \).

\( V_n(x) = \) Steady state probability that there are \( n(\geq 0) \) customers in the queue, the server is under vacation and the elapsed vacation time of the server is \( x \). Consequently \( V_n(x) = \int_0^\infty V_n(x)dx \) denotes the steady state probability that there are \( n \geq 0 \) customers in the queue and the server is under vacation irrespective of the value of \( x \).

Now the analysis of the limiting behavior of this queueing process at the stationary point of time can be done with the help of the following Kolmogorov forward equations

\[
\frac{d}{dx} P_n^{(1)}(x) + (\lambda + \mu_1(x)) P_n^{(1)}(x) = \lambda P_{n-1}^{(1)}(x), n = 1, 2, \ldots - \ldots (3.9)
\]

\[
\frac{d}{dx} P_0^{(1)}(x) + (\lambda + \mu_1(x)) P_0^{(1)}(x) = 0 - \ldots - \ldots - (3.10)
\]

\[
\frac{d}{dx} P_n^{(2)}(x) + (\lambda + \mu_2(x)) P_n^{(2)}(x) = \lambda P_{n-1}^{(2)}(x), n = 1, 2, \ldots - \ldots (3.11)
\]

\[
\frac{d}{dx} P_0^{(2)}(x) + (\lambda + \mu_2(x)) P_0^{(2)}(x) = 0 - \ldots - \ldots - (3.12)
\]

\[
\frac{d}{dx} P_n^{(3)}(x) + (\lambda + \mu_3(x)) P_n^{(3)}(x) = \lambda P_{n-1}^{(3)}(x), n = 1, 2, \ldots - \ldots (3.13)
\]

\[
\frac{d}{dx} P_0^{(3)}(x) + (\lambda + \mu_3(x)) P_0^{(3)}(x) = 0 - \ldots - \ldots - (3.14)
\]

\[
\frac{d}{dx} V_n(x) + (\lambda + \gamma(x)) V_n(x) = \lambda V_{n-1}(x), n = 1, 2, \ldots - \ldots - (3.15)
\]

\[
\frac{d}{dx} V_0(x) + (\lambda + \gamma(x)) V_0(x) = 0 - \ldots - \ldots - (3.16)
\]

Equations (3.9)-(3.16) are to be solved subject to the following boundary conditions:

\( P_0^{(1)}(0) = \int_0^\infty V_1(x)Y(x)dx + p \int_0^\infty P_0^{(2)}(x) \mu_2(x)dx + q \int_0^\infty P_0^{(3)}(x) \mu_3(x)dx \)

\[
+ q \int_0^\infty P_1^{(2)} \mu_2(x)dx + q \int_0^\infty P_1^{(3)} \mu_3(x)dx - - - - - (3.17)
\]
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\[ P_n^{(1)}(0) = \int_0^\infty V_{n+1}(x)Y(x)\,dx + p \int_0^\infty P_n^{(2)}(x)\,dx + p \int_0^\infty P_n^{(3)}(x)\,dx \]
\[ + q \int_0^\infty p_{n+1}^{(2)}(x)\,dx + q \int_0^\infty p_{n+1}^{(3)}(x)\,dx = - - - - - - (3.18) \]

\[ P_n^{(2)}(0) = q \int_0^\infty p_n^{(1)}(x)\,dx \quad n = 0,1 \ldots - - - - - - - - - (3.19) \]

\[ P_n^{(3)}(0) = q \int_0^\infty p_n^{(2)}(x)\,dx \quad n = 0,1 \ldots - - - - - - - - - (3.20) \]

\[ V_n(0) = 0, n = 1,2,3 \ldots - - - - - - - - - - - - - (3.21) \]

\[ V_0(0) = \int_0^\infty V_0(x)Y(x)\,dx + q \int_0^\infty p_0^{(2)}(x)\,dx + q \int_0^\infty p_0^{(3)}(x)\,dx = (3.22) \]

4. Steady State Probability Generating Functions of the Queue Size

**Theorem 1.** The system of Kolmogorov forward steady state equations to describe an M/G/1 feedback queue with three stages of service and with multiple server vacation is given by equations (3.9)-(3.22). Then the generating functions of the queue size are given by equations (4.35), (4.36) and (4.28) with respective initial conditions (4.30), (4.32) and (4.12).

**Proof.** We define the probability generating functions as follows.

\[ P^{(1)}(x, z) = \sum_{n=0}^\infty p_n^{(1)}(x)z^n, P^{(1)}(z) = \sum_{n=0}^\infty p_n^{(1)}z^n, |z| \leq 1, x > 0, - - - (4.1) \]

\[ P^{(2)}(x, z) = \sum_{n=0}^\infty p_n^{(2)}(x)z^n, P^{(2)}(z) = \sum_{n=0}^\infty p_n^{(2)}z^n, |z| \leq 1, x > 0, - - - (4.2) \]

\[ P^{(3)}(x, z) = \sum_{n=0}^\infty p_n^{(3)}(x)z^n, P^{(3)}(z) = \sum_{n=0}^\infty p_n^{(3)}z^n, |z| \leq 1, x > 0, - - - (4.3) \]
\[ V(x, z) = \sum_{0}^{\infty} V_n(x)z^n; \quad \sum_{0}^{\infty} V_n z^n, \quad |z| \leq 1, \quad x > 0 \quad \text{(4.4)} \]

We multiply both sides of equations (3.9) and (3.10) by suitable powers of \( z \), sum over \( n \) and use (4.1) and simplify. We thus have after algebraic simplifications

\[ \frac{d}{dx} P^{(1)}(x, z) + (\lambda - \lambda z + \mu_1(x))P^{(1)}(x, z) = 0 \quad \text{(4.5)} \]

Performing similar operations on equations (3.11) and (3.12) and using (4.2),

We have

\[ \frac{d}{dx} P^{(2)}(x, z) + (\lambda - \lambda z + \mu_2(x))P^{(2)}(x, z) = 0 \quad \text{(4.6)} \]

Similar operations on equations (3.13) and (3.14) and using (4.3)

\[ \frac{d}{dx} P^{(3)}(x, z) + (\lambda - \lambda z + \mu_3(x))P^{(3)}(x, z) = 0 \quad \text{(4.7)} \]

Similar operations on equations (3.15) and (3.16) yields

\[ \frac{d}{dx} V(x, z) + (\lambda - \lambda z + Y(x))V(x, z) = 0 \quad \text{(4.8)} \]

Next we multiply both sides of equation (3.17) by \( z \), multiply both sides of equation (3.18) by \( z^{n+1} \), sum over \( n \) from 1 to \( 1 \) , add the two results and use (4.1),(4.2),(4.3). Thus we obtain after mathematical adjustments

\[ zP_q^{-(1)}(0, z) = \int_{0}^{\infty} V(x, y)Y(x)dx + (q + pz)\int_{0}^{\infty} P^{(2)}(x, z)\mu_2(x)dx \]
\[ + (q + pz)\int_{0}^{\infty} P^{(3)}(x, z)\mu_3(x)dx - \int_{0}^{\infty} V_0(x, y)Y(x)dx \]
\[ - q\int_{0}^{\infty} P^{(2)}(x, z)\mu_2(x)dx - q\int_{0}^{\infty} P^{(3)}(x, z)\mu_3(x)dx \quad \text{(4.9)} \]
Performing similar operations to equations (3.19), (3.20), (3.21) and (3.22), we obtain

\[ p^{(2)}(0, z) = \int_0^\infty p^{(1)}(x, z) \mu_1(x) \, dx \]  

\[ p^{(3)}(0, z) = \int_0^\infty p^{(2)}(x, z) \mu_2(x) \, dx \]  

and \( V(0, z) = V_0(0) \)

Using equation (3.22) and (4.9), we now obtain

\[ zP_q^{-1}(0, z) = \int_0^\infty V(x, y)Y(x) \, dx + (q + pz) \int_0^\infty p^{(2)}(x, z) \mu_2(x) \, dx \]

\[ -V_0(0) + (q + pz) \int_0^\infty p^{(3)}(x, z) \mu_3(x) \, dx \]  

Integrating equations (4.5), (4.6), (4.7) and (4.8) between 0 and \( x \), we get

\[ p^{(1)}(x, z) = p^{(1)}(0, z) e^{-(\lambda - \lambda x) - \int_0^x \mu_1(t) \, dt} \]  

\[ p^{(2)}(x, z) = p^{(2)}(0, z) e^{-(\lambda - \lambda x) - \int_0^x \mu_2(t) \, dt} \]  

\[ p^{(3)}(x, z) = p^{(3)}(0, z) e^{-(\lambda - \lambda x) - \int_0^x \mu_3(t) \, dt} \]  

\[ V(x, z) = V(0, z) e^{-\int_0^x \mu_1(t) \, dt} \]  

Where \( p^{(1)}(0, z), p^{(2)}(0, z), p^{(3)}(0, z), and V(0, z) \) have been obtained in equations (4.13), (4.10), (4.11) and (4.12) respectively.

After integrating equation (4.14) with respect to \( x \), we have

\[ p^{(1)}(z) = p^{(1)}(0, z) \left[ \frac{1 - \frac{\bar{b}_1(\lambda - \lambda z)}{\lambda - \lambda z}}{\lambda - \lambda z} \right] \]
where \( \bar{B}_1(\lambda - \lambda z) = \int_0^\infty e^{-(\lambda - \lambda z)x} dB_1(x) \) is the Laplace transform of first stage of service time.

Now from equation (4.14) after some simplifications and using equation (2.2), we obtain
\[
\int_0^\infty P^{(1)}(x, z) \mu_1(x) \, dx = P^{(1)}(0, z) \bar{B}_1(\lambda - \lambda z) \tag{4.20}
\]

We now integrate equation (4.15) with respect to \( x \), to get
\[
P^{(2)}(z) = P^{(2)}(0, z) \left[ \frac{1 - \bar{B}_2(\lambda - \lambda z)}{\lambda - \lambda z} \right] \tag{4.21}
\]

Where \( \bar{B}_2(\lambda - \lambda z) = \int_0^\infty e^{-(\lambda - \lambda z)x} dB_2(x) \) is the Laplace transform of second stage of service time.

We see that by virtue of equation (4.15), we have
\[
\int_0^\infty P^{(2)}(x, z) \mu_2(x) \, dx = P^{(2)}(0, z) \bar{B}_2(\lambda - \lambda z) \tag{4.23}
\]

Integrate (4.16) with respect to \( x \), to get
\[
P^{(3)}(z) = P^{(3)}(0, z) \left[ \frac{1 - \bar{B}_3(\lambda - \lambda z)}{\lambda - \lambda z} \right] \tag{4.24}
\]

Where \( \bar{B}_3(\lambda - \lambda z) = \int_0^\infty e^{-(\lambda - \lambda z)x} dB_3(x) \) is the Laplace transform of third stage of service time.

By virtue of equation (4.16) we have
\[
\int_0^\infty P^{(3)}(x, z) \mu_3(x) \, dx = P^{(3)}(0, z) \bar{B}_3(\lambda - \lambda z) \tag{4.26}
\]
After integrating equation (4.13), we get
\[ V(z) = V(0, z) \left[ 1 - \bar{V}(\lambda - \lambda z) \right] - - - - - - (4.27) \]

And therefore by virtue of equation (4.13), we get
\[ \int_0^\infty V(x, z)Y(x)dx = V(0, z)\bar{V}(\lambda - \lambda z) - - - - - - (4.28) \]

Where
\[ \bar{V}(\lambda - \lambda z) = \int_0^\infty e^{-(\lambda - \lambda z)x} dV(x) - - - - - - (4.29) \]

is the Laplace-Stieltjes transform of the vacation time.

Further, using equations (4.10),(4.11),(4.12),(4.20),(4.23),(4.26) and (4.28) into equation (4.10) and after some simplifications, we obtain
\[ P^{(1)}(0, z) = \frac{\bar{V}(\lambda - \lambda z) - 1}{DR} V_0(0) - - - - - - (4.30) \]

Now from equations (4.10) and (4.20), we get
\[ P^{(2)}(0, z) = P^{(1)}(0, z) \bar{B}_1(\lambda - \lambda z) - - - - - - (4.31) \]

From equation (4.30), the above equation reduces to
\[ P^{(2)}(0, z) = \frac{\bar{B}_1(\lambda - \lambda z) \bar{B}_2(\lambda - \lambda z) \bar{V}(\lambda - \lambda z) - 1}{DR} V_0(0) - - - - - - (4.32) \]

Now from equation (4.11) and (4.26) we get
\[ P^{(3)}(0, z) = P^{(3)}(0, z) \bar{B}_2(\lambda - \lambda z) - - - - - - (4.33) \]

From equation (4.30),(4.31),(4.32) above equation reduces to
\[ P^{(3)}(0, z) = \frac{\bar{B}_1(\lambda - \lambda z) \bar{B}_2(\lambda - \lambda z) \bar{V}(\lambda - \lambda z) - 1}{DR} V_0(0) - - - - - - (4.34) \]
Now using equation (4.30) into equation (4.18), we have

\[
P^{(1)}(z) = \frac{\bar{V}(\lambda - \lambda z) - 1}{DR} V_0(0) \left[ \frac{1 - B_1(\lambda - \lambda z)}{\lambda - \lambda z} \right] = - - - -(4.35)
\]

By using equation (4.32), equation (4.21) can be written as

\[
P^{(2)}(z) = \frac{\bar{B}_1(\lambda - \lambda z)\bar{V}(\lambda - \lambda z) - 1}{DR} V_0(0) \left[ \frac{1 - B_2(\lambda - \lambda z)}{\lambda - \lambda z} \right] = - - - -(4.36)
\]

By using equation (4.34), equation (4.24) can be written as

\[
P^{(3)}(z) = \frac{\bar{B}_1(\lambda - \lambda z)\bar{B}_2(\lambda - \lambda z) \bar{V}(\lambda - \lambda z) - 1}{DR} V_0(0) \left[ \frac{1 - B_3(\lambda - \lambda z)}{\lambda - \lambda z} \right] = - - - -(4.37)
\]

\[\text{Where } DR = z - (q + pz)\bar{B}_1(\lambda - \lambda z)\bar{B}_2(\lambda - \lambda z) - (q + pz)\bar{B}_1(\lambda - \lambda z)\bar{B}_2(\lambda - \lambda z)\bar{B}_3(\lambda - \lambda z)\]

Also from equation (4.27), we get

\[V(z) = V_0(0) \left[ \frac{1 - \bar{V}(\lambda - \lambda z)}{\lambda - \lambda z} \right] = - - - -(4.38)\]

Thus \(P^{(1)}(z), P^{(2)}(z), P^{(3)}(z)\) and \(V(z)\) can be determined from equations (4.38),(4.35),(4.36),(4.37) and (4.38)

where \(P^{(1)}(0, z), P^{(2)}(0, z), P^{(3)}(0, z)\) and \(V(0, z)\) are given by equations (4.30),(4.32),(4.34) and(4.12) respectively.

In order to determine \(P^{(1)}(z), P^{(2)}(z), P^{(3)}(z)\) and \(V(z)\) completely, we have yet to determine the only unknown \(V_0(0)\) which appears in the numerators of the right hand sides of equations (4.35),(4.36),(4.37) and (4.38) respectively. For that purpose, we shall use the normalizing condition

\[
P^{(1)}(1) + P^{(2)}(1) + P^{(3)}(1) + V(1) = 1 - - - -(4.39)\]
THEOREM 2. The steady state probabilities for an M/G/1 feedback queue with three stages of service and with multiple server vacation is given by

\[ P^{(1)}(1) = \left[ \frac{\lambda E(v_1)E(v)}{q - 2\lambda E(v_1) - 2\lambda E(v_2) - \lambda E(v_3) - P} \right] V_0(0) \]

\[ P^{(2)}(1) = \left[ \frac{\lambda E(v_2)E(v)}{q - 2\lambda E(v_1) - 2\lambda E(v_2) - \lambda E(v_3) - P} \right] V_0(0) \]

\[ P^{(3)}(1) = \left[ \frac{\lambda E(v_3)E(v)}{q - 2\lambda E(v_1) - 2\lambda E(v_2) - \lambda E(v_3) - P} \right] V_0(0) \]

\[ V(1) = E(v)V_0(0) \]

Where \( P^{(1)}(1), P^{(2)}(1), P^{(3)}(1), \) and \( V(1) \) respectively denote the steady state probabilities that the server is providing first stage, second stage and third stage of service and that the server is on vacation respectively without regard to the number of customers in the queue.

\[ V_0(0) = \frac{q - p - 2\lambda E(v_1) - 2\lambda E(v_2) - \lambda E(v_3)}{\lambda E(v) \left( E(v_1) + E(v_2) + E(v_3) + \frac{1}{\lambda} \right)} \]

Proof. We have from equations (4.35),(4.36),(4.37) and (4.38)

\[ P^{(1)}_q(z) = \frac{\bar{V}(\lambda - \lambda z) - 1}{DR} V_0(0) \left[ \frac{1 - \bar{B}_1(\lambda - \lambda z)}{\lambda - \lambda z} \right] \]

\[ P^{(2)}_q(z) = \frac{(\bar{B}_1(\lambda - \lambda z))(\bar{V}(\lambda - \lambda z) - 1)}{DR} V_0(0) \left[ \frac{1 - \bar{B}_2(\lambda - \lambda z)}{\lambda - \lambda z} \right] \]

\[ P^{(3)}_q(z) = \frac{(\bar{B}_1(\lambda - \lambda z))(\bar{B}_2(\lambda - \lambda z))(\bar{V}(\lambda - \lambda z) - 1)}{DR} V_0(0) \left[ \frac{1 - \bar{B}_3(\lambda - \lambda z)}{\lambda - \lambda z} \right] \]
\[ V(z) = V_0(0) \left[ \frac{1 - \bar{V} (\lambda - \lambda z)}{\lambda - \lambda z} \right] \quad (4.48) \]

Where \( DR = z - (q + pz) \bar{B}_1(\lambda - \lambda z) \bar{B}_2(\lambda - \lambda z) \)
\[ - (q + pz) \bar{B}_1(\lambda - \lambda z) \bar{B}_2(\lambda - \lambda z) \bar{B}_3(\lambda - \lambda z) \]

However, since the right sides of each of the above equations are indeterminate of the \( 0/0 \) form at \( z=1 \), we resort to use the use of L'Hôpital's rule and obtain on simplifying

\[ p^{(1)}(1) = \left[ \frac{\lambda E(v_1)E(v)}{q - 2\lambda E(v_1) - 2\lambda E(v_2) - \lambda E(v_3) - p} \right] V_0(0) \quad (4.49) \]

\[ p^{(2)}(1) = \left[ \frac{\lambda E(v_2)E(v)}{q - 2\lambda E(v_1) - 2\lambda E(v_2) - \lambda E(v_3) - p} \right] V_0(0) \quad (4.50) \]

\[ p^{(3)}(1) = \left[ \frac{\lambda E(v_3)E(v)}{q - 2\lambda E(v_1) - 2\lambda E(v_2) - \lambda E(v_3) - p} \right] V_0(0) \quad (4.51) \]

\[ V(1) = E(v)V_0(0) \quad (4.52) \]

Where \( E(v_1), E(v_2), E(v_3) \) and \( E(v) \) denote the mean service time of first stage service, second stage of service, third stage of service, and mean vacation time respectively. Note that the results (4.49)-(4.52) give the steady state probabilities that the server is providing first stage of service, second stage of service, third stage of service, and under vacation respectively.

Now using equations (4.49)-(4.52) into the normalizing condition (4.34) and simplifying, we obtain

\[ V_0(0) = \frac{q - p - 2\lambda E(v_1) - 2\lambda E(v_2) - \lambda E(v_3)}{\lambda E(v) \left( E(v_1) + E(v_2) + E(v_3) + \frac{1}{\lambda} \right)} \quad (4.53) \]

Also from equation (4.53), we obtain system’s utilization factor.
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\[ \rho = \frac{2E(v_3) + 3E(v_2) + 3E(v_1) + \frac{2p}{\lambda}}{E(v_1) + E(v_2) + E(v_3) + \frac{1}{\lambda}} \]  

(4.54)

Where \( \rho < 1 \) is the stability condition under which the steady state exists.

Thus we have now explicitly determined all the steady state probability generating functions \( p_q^{(1)}(z), p_q^{(2)}(z), p_q^{(3)}(z) \) and \( V(z) \) of the queue.

5. The Mean Number in the system

Now, we define \( P_q(z) \) as the probability generating function of the queue size. Then we have

\[ P_q(z) = V_q(z) + z(P_1(z) + P_2(z) + P_3(z)) \]  

(5.1)

Let \( L_q \) and \( L \) denote the steady state average queue size and system size respectively. We have

\[ L_q = \lim_{z \to 1} \frac{d}{dz} P_q(z) = P_q'(1) = \lim_{z \to 1} \frac{D'(z)N''(z) - N'(z)D''(z)}{2(D'(z))^2} \left[ \frac{V_0(0)}{\lambda} \right] \]  

(5.2)

Where primes and double primes in equation (5.2) denote the first and second derivatives at \( z = 1 \). Using equations (4.45) to (4.48) into equation (5.1), we have

\[ P_q(z) = \frac{N(z)}{D(z)} \left[ \frac{V_0(0)}{\lambda} \right] \]  

(5.3)

Where

\[ N(z) = [\tilde{V}(\lambda - \lambda z) - 1][qB_1(\lambda - \lambda z)B_2(\lambda - \lambda z)B_3(\lambda - \lambda z)] \]  

(5.4)

\[ D(z) = z - (q + pz)B_1(\lambda - \lambda z)B_2(\lambda - \lambda z) - (q + pz)B_1(\lambda - \lambda z)B_2(\lambda - \lambda z)B_3(\lambda - \lambda z) \]  

(5.5)

Carrying out the derivatives at \( z = 1 \), we have

\[ N'(1) = 2\lambda E(v)q \]  

(5.6)

\[ N''(1) = q2\lambda^2 \left[ E(v^2) + 2\frac{E(v)}{\mu} + 2E(v)E(v_2) + E(v)E(v_3) \right] \]  

(5.7)

\[ D'(1) = q - p - 2\lambda \left[ \frac{1}{\mu} + E(v_2) \right] - \lambda E(v_3) \]  

(5.8)
Where \( E(v_1^2), E(v_2^2), E(v_3^2) \) and \( E(v^2) \) denote the second moments of the first stage of service, second stage of service, third stage of service, and vacation time respectively. Using equations (5.6)-(5.9) into equation (5.2), we have obtained \( L_q \) in closed form. Further, we find the average system size \( L \) using Little's formula. Thus we have

\[
L = L_q + \rho \quad \cdots \quad (5.10)
\]

Where \( L_q \) has been found in equation (5.2) and \( \rho \) is obtained from equation (4.54).

6. The Mean Waiting Time

Let \( W_q \) and \( W \) denote the mean waiting time in the queue and the system respectively. Then using Little's formula, we obtain

\[
W_q = \frac{L_q}{\lambda} \quad \cdots \quad (6.1)
\]

7. Special cases

7.1. Case 1: First Stage of Service Follow Exponential Distribution

In this case, we put \( \bar{B}_1(\lambda - \lambda z) = \frac{\mu_1}{\lambda - \lambda z + \mu_1}, E(v_1) = \frac{1}{\mu_1}, E(v_1^2) = \frac{2}{\mu_1^2} \) in the main results and obtain

\[
P_q(z) = \frac{V_1(\lambda - \lambda z) - 1 \left( q \left( \frac{\mu_1}{\lambda - \lambda z + \mu_1} \right) B_1(\lambda - \lambda z) + q \left( \frac{\mu_1}{\lambda - \lambda z + \mu_1} \right) B_2(\lambda - \lambda z) B_3(\lambda - \lambda z) \right)}{z - (q + p)q \left( \frac{\mu_1}{\lambda - \lambda z + \mu_1} \right) B_2(\lambda - \lambda z) - (q - p)z \left( \frac{\mu_1}{\lambda - \lambda z + \mu_1} \right) B_2(\lambda - \lambda z) B_3(\lambda - \lambda z)} \times \left[ \frac{V_0(0)}{\lambda} \right] \quad \cdots \quad (7.1)
\]

And \( L_q \) is given by (5.2), where

\[
N'(1) = 2\lambda E(v)q \quad \cdots \quad (7.2)
\]

\[
N''(1) = q2\lambda^2 \left[ E(v^2) + 2E(v)E(v_2) + E(v)E(v_3) \right] \quad \cdots \quad (7.3)
\]
$D'(1) = q - p - 2\lambda \left[ \frac{1}{\mu_1} + E(v_2) \right] - \lambda E(v_3) - - - - (7.4)$

$D''(1) = -\left\{ 2p\lambda \left[ \frac{1}{\mu_1} + 2E(v_2) + E(v_3) \right] + 2\lambda^2 \left[ \frac{2}{\mu_1^2} + E(v_2^2) + \frac{2}{\mu_1} E(v_2) + \frac{E(v_3)}{\mu_1} \right] + \lambda^2 \left[ E(v_2^2) + E(v_2)E(v_3) + pE(v_2)E(v_3) \right] \right\} - - - - (7.5)$

In addition, $L, W$ and $W_q$ for this case can also be found from the main results.

### 7.2 Case 2; All the Three Services are Exponential

In this case, we let

$\bar{B}_2(\lambda - \lambda z) = \frac{\mu_1}{\lambda - \lambda z + \mu_2}, E(v_2) = \frac{1}{\mu_2}, E(v_2^2) = \frac{2}{\mu_2^2}, \bar{B}_3(\lambda - \lambda z) = \frac{\mu_1}{\lambda - \lambda z + \mu_3}$.

$E(v_3) = \frac{1}{\mu_3}, E(v_3^2) = \frac{2}{\mu_3^2}$ in all the results obtained in Case 1.

$P_q(z) = \left[ \frac{\bar{V}_1(\lambda - \lambda z)^{-1}}{z - q + pz} \right] - - - - - - - - (7.6)$

And $L_0$ is given by (5.2), when

$N'(1) = 2\lambda E(v)q - - - - - - - - (7.7)$

$N''(1) = q2\lambda^2 \left[ E(v^2) + 2E(v) + \frac{E(v)}{\mu_1} + 2E(v) + \frac{E(v)}{\mu_2} + E(v) + \frac{E(v)}{\mu_3} \right] - - - - (7.8)$

$L, W$ and $W_q$ can also be found from the main results.

### 7.3 Case 3; Vacation Period following Exponential Distribution;

Let $\bar{V}(\lambda - \lambda z) = \frac{y}{\lambda - \lambda z + y}, E(v) = \frac{1}{y}, E(v^2) = \frac{1}{y^2}$ in all the results obtained in Case 2.

### 7.4 Case 4; No Server Vacation;

Here, we consider $[\bar{V}(\lambda - \lambda z) - 1] - 1$ in equations (5.3) & obtain

$P_q(z) = \frac{N(z)}{D(z)} \left[ \frac{V_0(0)}{\lambda} \right] - - - - (7.11)$
Where
\[ N(z) = [z - 1][q\bar{B}_1(\lambda - \lambda z)\bar{B}_2(\lambda - \lambda z) + q\bar{B}_1(\lambda - \lambda z)\bar{B}_2(\lambda - \lambda z)\bar{B}_3(\lambda - \lambda z)] \quad (7.12) \]
\[ D(z) = z - (q + pz)\bar{B}_1(\lambda - \lambda z)\bar{B}_2(\lambda - \lambda z) - (q + pz)\bar{B}_1(\lambda - \lambda z) \]
\[ \bar{B}_2(\lambda - \lambda z)\bar{B}_3(\lambda - \lambda z) \quad (7.13) \]

Further $L_q$, $L$, $W_q$ & $W$ can be derived from the results obtained in previous sections.

8. Conclusion

This paper clearly analyses the transient solution, steady state results and the various performance measures of the queueing system with three stages of service with multiple server vacation, server provides essential service in all the stages to the arriving customers. This model can be utilized in large scale manufacturing industries and communication networks.

References


[13] J. Medhi, A single server poisson input queue with a second optional service,


[16] K.C. Madan, A single channel queue with bulk service subject to interruptions,


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