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Some Generalized Trigonometric Sine Functions and Their Applications

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Abstract

In this paper, it is shown that D. Shelupsky's generalized sine function, and various general sine functions developed by P. Drábek, R. Manásevich and M. Ôtani, P. Lindqvist, including the generalized Jacobi elliptic sine function of S. Takeuchi can be defined by systems of first order nonlinear ordinary differential equations with initial conditions. The structure of the system of differential equations is shown to be related to the Hamilton System in Lagrangian Mechanics. Numerical solutions of the ODE systems are solved to demonstrate the sine functions graphically. It is also demonstrated that the some of the generalized sine functions can be used to obtain analytic solutions to the equation of a nonlinear spring-mass system.

Keywords: generalized sine, Hamilton system, nonlinear spring, vibration, analytic solution

1 Introduction

In this work, we show that the various generalized trigonometric sine functions found in the work of D. Shelupsky, P. Drábek and R. Manásevich and M. Ôtani, P. Lindqvist, and the ones including the generalized Jacobi elliptic sine function of S. Takeuchi all can be defined by systems of first order ordinary differential equations (ODE's) subject to initial conditions. The connections between these generalized sine functions and their associated generalized constant Pi are demonstrated in section II. The connections to the ODE's are demonstrated in Section III. Linking the ODE's to the Hamilton System in Lagrangian Mechanics with an application for solutions to a nonlinear spring-mass equation are demonstrated in IV. The ODE's allow us to numerically calculate the various sine functions with a uniform approach and numerical presentations of the sine functions are presented in section V.

2 Connections between the generalized sine functions

The traditional Euclidean sine and cosine functions can be defined as the solution of a first order differential equation system with initial condition:

$$\begin{cases} y' = x, \quad y(0) = 0 \\ x' = -y, \; x(0) = 1 \end{cases}$$

which is equivalent to the following system of second order equations:

$$\begin{cases} y'' + y = 0, \; y(0) = 0, \; y'(0) = 1 \\ x'' + x = 0, \; x(0) = 1, \; x'(0) = 0 \end{cases}$$

Since

$$yy' + xx' = yx + (-xy) = 0$$

we have

$$\frac{d}{dt}(\frac{1}{2}x^2 + \frac{1}{2}y^2) = 0$$

and

$$x^2 + y^2 = 1$$

that is the Pythagorean identity

$$|\sin t|^2 + |\cos t|^2 = 1$$

where $\sin t = y(t)$ and $\cos t = x(t)$. The sine and cosine functions can also be defined by

$$\sin^{-1} t \doteq \begin{cases} \int_0^t \frac{1}{\sqrt{1-s^2}} ds, 0 \le t \le 1 \\ -\int_0^{-t} \frac{1}{\sqrt{1-s^2}} ds, -1 \le t \le 0 \end{cases}$$

and the Euclidean number

$$\pi \doteq 2\sin^{-1}1 = 2\int_0^1 \frac{1}{\sqrt{1-s^2}}ds$$

equals the bounded area of the unit circle enclosed by $x^2 + y^2 = 1$. David Shelupsky's definition of a generalized sine function [1] is based on the differential equation approach. First, let

$$\phi(\varepsilon) = |\varepsilon|^{p-2}\varepsilon,\ p \in R^+$$

be the power-law function, then let $\sin_p t = y(t)$ and $\cos_p t = x(t)$, where $x(t)$ and $y(t)$ are the solutions to the nonlinear initial value problem:

$$\begin{cases} y' = \phi(x),\ y(0) = 0 \\ x' = -\phi(y),\ x(0) = 1 \end{cases}$$

Like the linear counterpart, we have

$$\begin{cases} \phi(y)y' = \phi(y)\phi(x) \\ \phi(x)x' = -\phi(y)\phi(x) \end{cases}$$

which gives

$$\phi(y)y' + \phi(x)x' = 0$$

and

$$|x(t)|^p + |y(t)|^p = 1$$

The above equation is the generalized Pythagorean identity:

$$|\sin_p(t)|^p + |\cos_p(t)|^p = 1$$

The system is also equivalent to the following two second order ODE's with boundary conditions:

$$\begin{cases} [\phi^{-1}(y')]' + \phi(y) = 0, y(0) = 0, y'(0) = 1 \\ [\phi^{-1}(x')]' + \phi(x) = 0, x(0) = 1, y'(0) = 0 \end{cases}$$

Similar to the case when $p = 2$, the inverse sine function has the following integral form:

$$\sin_p^{-1} t \doteq \begin{cases} \int_0^t \frac{1}{\sqrt[p]{(1-s^p)^{p-1}}}ds, 0 \le t \le 1 \\ -\int_0^{-t} \frac{1}{\sqrt[p]{(1-s^p)^{p-1}}}ds, -1 \le t \le 0 \end{cases}$$

and

$$|\sin t|^p + |\cos t|^p = 1$$

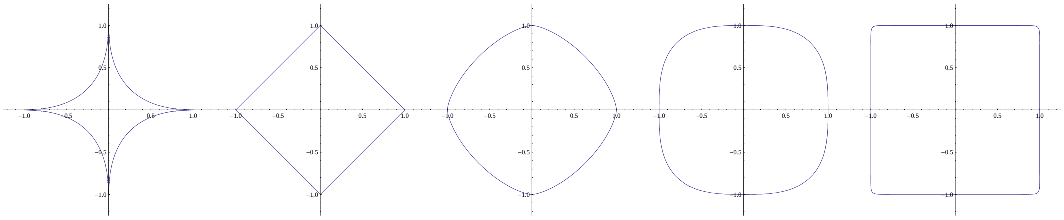
A generalized Pi is defined by David Shelupsky [1] is

$$\pi_p \doteq 2 \int_0^1 \frac{1}{\sqrt[p]{(1-s^p)^{p-1}}} ds$$

which is shown to be the bounded area enclosed by the graph of $|x|^p + |y|^p = 1$, and it has the following interesting property

$$\Gamma(\tfrac{1}{2}) = 2^{\frac{3}{4}} \sqrt{\pi_4 \sqrt{\frac{1}{2}\pi_2}}, \quad \Gamma(\tfrac{1}{8}) = 2^{\frac{7}{8}} \sqrt{2\pi_8 \sqrt{\pi_4 \sqrt{\frac{1}{2}\pi_2}}}$$

Several values of the generalized Pi are given in Table 1: For more numerical values of π_p see [2]. It is also interesting to look at the graphs of $|x|^p + |y|^p = 1$ for various values of p in Fig. 1.

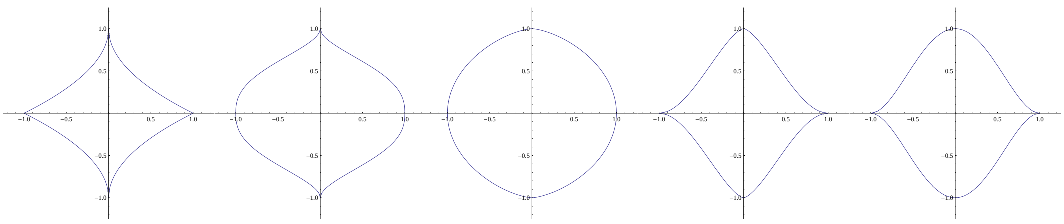


(a) $p = 0.5$ (b) $p = 1$ (c) $p = 1.5$ (d) $p = 2.5$ (e) $p = 30$

Figure 1: Plots for $|x|^p + |y|^p = 1$ with different p.

Table 1: Values of π

π_1	π_2	π_3
2.000	3.1415	3.595



(a) $p = .5, q = 1$ (b) $p = .5, q = 2.5$ (c) $p = 1.5, q = 2$ (d) $p = 1.5, q = .5$ (e) $p = 2, q = .5$

Figure 2: Plots for $|x|^p + |y|^q = 1$ with different p and q.

A similar version of the generalized sine function is developed by M. Del Pino and M. Elgueta and R. Manasevich by the following:

$$\sin_p^{-1} t \doteq \begin{cases} \int_0^t \frac{1}{\sqrt[p]{1-\frac{s^p}{p-1}}} ds, 0 \le t \le 1 \\ -\int_0^{-t} \frac{1}{\sqrt[p]{1-\frac{s^p}{p-1}}} ds, -1 \le t \le 0 \end{cases}$$

for which the generalized Pi is defined by

$$\pi_p \doteq 2 \int_0^{(p-1)^{1/p}} \frac{1}{\sqrt[p]{1-\frac{s^p}{p-1}}} ds.$$

This sine function is used for the solution of the following eigenvalue value problem [3].

$$\left(|u'|^{p-2}u'\right)' + \lambda |u|^{p-2}u = 0$$

satisfying

$$u(0) = 0, u'(0) = \alpha$$

A widely cited version of generalized sine is defined by Peter Lindqvist [4], in which the inverse sine is defined by

$$\sin_p^{-1} t \doteq \begin{cases} \int_0^t \frac{1}{\sqrt[p]{1-s^p}} ds, 0 \le t \le 1 \\ -\int_0^{-t} \frac{1}{\sqrt[p]{1-s^p}} ds, -1 \le t \le 0 \end{cases}$$

for which the generalized Pi is defined by

$$\pi_p \doteq 2 \int_0^1 \frac{1}{\sqrt[p]{1-s^p}} ds.$$

It is interesting that we also have $|\sin t|^p + |\cos t|^p = 1$, and $u = \sin_p t$ is the unique solution to the boundary value problem:

$$\left(|u'|^{p-2}u'\right)' + (p-1)|u|^{p-2}u = 0$$

satisfying

$$u(0) = 0, \quad u'(0) = 1$$

The above sine functions are depending upon one parameter p. The definition of a two parameter generalized sine by Lindqvist [5], and Drábek and Manásevich [6] is given by the integral of the inverse sine:

$$\sin_{pq}^{-1}(t) \doteq \begin{cases} \int_o^t \frac{ds}{\sqrt[p]{1-s^q}}, 0 \le t \le 1 \\ -\int_o^{-t} \frac{ds}{\sqrt[p]{1-s^q}}, -1 \le t \le 0 \end{cases}$$

The corresponding generalized Pi is defined by

$$\pi_{pq} = 2\sin_{pq}^{-1}(1) = \frac{2}{q}B(\frac{1}{q}, 1-\frac{1}{p})$$

where $B(\frac{1}{q}, 1-\frac{1}{p})$ is the Euler Beta function evaluated at $(\frac{1}{q}, 1-\frac{1}{p})$ (see also the work of [7]). It is proved by Shingo Takeuchi [8] that for $p > 1, q > 1$ this generalized sine function is the solution of the following second-order differential equation

$$\frac{p}{p-1}\left(|u'|^{p-2}u'\right)' + q|u|^{q-2}u = 0$$

satisfying

$$u(0) = 0, u'(0) = 1$$

and that the identity

$$|\sin_{pq}(t)|^p + |\cos_{pq}(t)|^q = 1$$

holds, in which

$$\cos_{pq}(t) = (1 - \sin_{pq}^q(t))^{\frac{1}{p}}$$

This sine function is more general since both the previous mentioned ones are special cases of this one. It is widely used in the literature for studying the eigenvalues and eigenfunctions associated with the p-Laplacian operator, see e.g., [4, 5, 9, 10]. Since the list of references for this type of work is long, for the interest of brevity we only list a few here. This definition is consistent with the integral definition found in the work of [11, 12].

We now demonstrate that the above generalized sine functions and the elliptic Jacobi sine functions can be treated as special cases in the unifying framework of a single system of first order differential equations with an initial condition and to show that under this frame work, the above mentioned various generalized sine functions are naturally connected to the solutions of free vibrations systems in nonlinear Hamiltonian Mechanics.

3 The general system of equations that defines sine functions

The framework for defining generalized sine functions and for calculating their analytic forms is motivated by the structure of the Hamilton equations in Lagrangian Mechanics. For this purpose, the following more general system of first order ODE with initial condition is considered:

$$\begin{cases} y' = \phi(x), \ y(0) = 0 \\ x' = -\psi(y), \ x(0) = 1 \end{cases} \tag{1}$$

where ϕ and ψ are suitable functions satisfying appropriate conditions to guarantee the existence and uniqueness of the solution of Eq. (1). Assuming that ϕ and ψ are integrable and denote the Φ and Ψ anti-derivatives of them respectively satisfying $\Phi(0) = \Psi(0) = 0$. Since

$$\psi(y)y' + \phi(x)x' = 0$$

we have

$$\Psi(y(t)) + \Phi(x(t)) = \Phi(1)$$

and

$$\Psi(y(t)) + \Phi(\phi^{-1}(y'(t))) = \Phi(\phi^{-1}(1))$$

Symbolically, the solution $y(t)$ of Eq. (1) can be defined in terms of an integral equation

$$\int_{y(0)}^{y(t)} \frac{dy}{\phi[\Phi^{-1}(\Phi(\phi^{-1}(1)) - \Psi(y)]} = t$$

provided that the appropriate definitions of Φ^{-1} and ϕ^{-1} are available.

If the integral function

$$f(y) = \int_0^y \frac{d\tau}{\phi[\Phi^{-1}(\Phi(\phi^{-1}(1)) - \Psi(\tau))]}$$

is well-defined for

$$0 \le \psi(y) \le \Phi(\phi^{-1}(1))$$

then the generalized sine function $\sin_{\phi,\psi} t$ can be defined as the solution

$$y(t) = f^{-1}(t), \text{ for } 0 \le t \le \frac{\pi_{\phi,\psi}}{2}$$

where

$$\pi_{\phi,\psi} \doteq \int_0^1 \frac{dy}{\phi[\Phi^{-1}(\Phi(\phi^{-1}(1)) - \Psi(y)]}$$

is the corresponding generalized Pi. The standard technique of odd-extension of functions on a finite domain to periodic functions on the whole real line can be used to extend this generalized sine as an odd periodic function of $2\pi_{\phi,\psi}$. The generalized cosine is defined as $\cos_{\phi,\psi} t = x(t)$, and the following generalized Pythagorean identity

$$\Psi(\sin_{\phi,\psi} t) + \Phi(\cos_{\phi,\psi} t) = \Phi(\phi^{-1}(1))$$

holds. In particular, if

$$\phi(\varepsilon) = p|\varepsilon|^{p-2}\varepsilon, \ \psi(\varepsilon) = q|\varepsilon|^{q-2}\varepsilon, \ p, q \in R^+$$

by defining

$$\sin_{p,q}t = \sin_{\phi,\psi}t, \quad \cos_{p,q}t = \cos_{\phi,\psi}t$$

the identity

$$|\sin_{p,q}(t)|^p + |\cos_{p,q}(t)|^q = 1$$

follows. Also, we have

$$\pi_{p,q} = 2\int_0^1 \frac{1}{\sqrt[p]{1-s^q}}ds = \frac{2}{q}B(\frac{1}{q}, 1-\frac{1}{p})$$

Especially, for $p = q$, $\sin_{p,p}(t) \equiv \sin_p(t)$ given by the definition of David Shelupsky [1]. In another case, if

$$\phi(\varepsilon) = \sqrt[p]{\frac{p-1}{p}}|\varepsilon|^{\frac{2-p}{p-1}}\varepsilon, \quad \psi(\varepsilon) = q|\varepsilon|^{q-2}\varepsilon,\ p > 1, q > 1$$

then

$$\sin_{\phi,\psi}(t) \equiv \sin_{pq}(t)$$

which is the sine function given by the definition of Drábek and Manásevich [6], and M. Ôtani [7]. Figure 2 shows graphs of $|y(t)|^p + |x(t)|^q = 1$ for various values of p, q, and π_{pq} is the area of the bounded region enclosed.

As the third special case, let

$$\psi(\varepsilon) = \frac{q(p-1)}{p}(1 + k^q(1 - 2|\varepsilon|^q)|\varepsilon|^{q-2}\varepsilon,\ p > 1, q > 1$$

and

$$\phi(\varepsilon) = |\varepsilon|^{p-2}\varepsilon,\ p > 1$$

Then we have

$$\sin_{\phi,\psi}(t) = sn_{pq}(t, k)$$

where $sn_{pq}(t, k)$ is well-known generalized Jacobi elliptic sine function. It is interesting to note that the connection between $sn_{pq}(t, k)$ and $\sin_{pq}(t)$ is presented by [8] as

$$\sin_{\phi,\psi}(t) \equiv \sin_{pq}(am_{pq}(t, k))$$

where the amplitude function $am_{pq}(t, k)$ is defined by

$$t = \int_0^{am_{pq}(t,k)} \frac{1}{\sqrt[p]{1 - k^q {\sin_{pq}}^q(s)}}ds$$

4 Applications of the generalized sine functions

In fact, the symbolic definition presented by Eq. (1) is connected to the Hamiltonian system of Lagrangian Mechanics. The well-known Hamilton system is given by

$$\begin{cases} \frac{dp}{dt} = -\frac{\partial H(p,q,t)}{\partial q} \\ \frac{dq}{dt} = \frac{\partial H(p,q,t)}{\partial p} \end{cases} \tag{2}$$

where $H = T + V$ is the Hamilton energy function, T is the kinetic energy, and V is the potential energy. Since

$$\frac{\partial H(p,q,t)}{\partial p}\frac{dp}{dt} + \frac{\partial H(p,q,t)}{\partial q}\frac{dq}{dt} = 0$$

We have $H(p, q, t) = C$, which is the equation of conservation of energy. It is obvious that Eq. (2) is an important case of Eq. (1).

As an example of applications of the generalized sine functions, consider the following equation of motion for a nonlinear spring-mass system:

$$M\ddot{X} + K|X|^{n-1}X = 0$$

This system is subject to the following generalized Hooke's law:

$$F = -K|X|^{n-1}X, n > 0$$

where F is the restoring force, M is the mass, K is the Hooke's constant, n is the work-hardening index in the theory of plastic mechanics, and X is the displacement of the mass of the equilibrium position. The dot derivative $\dot{X}$ stands for the time derivative of X or the velocity of the mass. The Hamilton function for this example is

$$H(p,q) = \frac{p^2}{2} + V(q) = \frac{p^2}{2} + \frac{K|q|^{n+1}}{(n+1)M}$$

where $p = \dot{X}$ and $q = X$. The corresponding Hamilton system is

$$\begin{cases} \frac{dx}{dt} = -\frac{K|y|^{n-1}y}{M} \\ \frac{dy}{dt} = x \end{cases}$$

where $y = q$, and $x = p$. For the initial condition $y(0) = 0, x(0) = 1$, we have the following solution

$$\begin{cases} y(t) = \sin_{\phi,\psi}(t) \\ x(t) = \cos_{\phi,\psi}(t) \end{cases}$$

where $\phi(\varepsilon) = \frac{|\varepsilon|^{n-1}\varepsilon}{K}$ and $\psi(\varepsilon) = \frac{\varepsilon}{M}$. The identity

$$\frac{K|\sin_{\phi,\psi}(t)|^{n+1}}{M(n+1)} + \frac{|\cos_{\phi,\psi}(t)|^2}{2} = \frac{1}{2}$$

is satisfied. The specific expressions of the solution of the above system with initial condition $y(0) = 0, x(0) = y'(0) = \dot{X}$ can be derived by solving

$$\frac{|\dot{y}|^2}{2} + \frac{K|y|^{n+1}}{(n+1)M} = \frac{|\dot{y}(0)|^2}{2}$$

From

$$\frac{dy}{dt} = \pm\sqrt{[\dot{y}(0)]^2 - \frac{2K}{M(n+1)}|y|^{n+1}}$$

we get

$$\frac{dy}{\sqrt{[\dot{y}(0)]^2 - \frac{2K}{M(n+1)}|y|^{n+1}}} = \pm dt$$

Let

$$s = \sqrt[n+1]{\frac{2K}{M(n+1)[\dot{y}(0)]^2}}y$$

then

$$\frac{ds}{\sqrt{1 - s^{n+1}}} = \pm|\dot{y}(0)|\sqrt[n+1]{\frac{2K}{M(n+1)[\dot{y}(0)]^2}}dt$$

and we have

$$\int_0^{\frac{y(t)}{A}} \frac{ds}{\sqrt{1 - s^{n+1}}} = \pm\omega t$$

where

$$\frac{1}{A} = \sqrt[n+1]{\frac{2K}{(n+1)M|\dot{y}(0)|^2}}, \quad \omega = \sqrt[n+1]{\frac{2K|\dot{y}(0)|^{n-1}}{(n+1)M}}$$

Therefore, since $y = X$, the solution of the vibration system can be expressed in terms of the generalized sine defined by Eq. (1)

$$X(t) = A\sin_{\phi,\psi}(\omega t)$$

where

$$\phi(\varepsilon) = \varepsilon, \psi(\varepsilon) = \frac{K|\varepsilon|^{q-2}\varepsilon}{M}, A = \sqrt[n+1]{\frac{(n+1)M|\dot{X}(0)|^2}{2K}}, \omega = \sqrt[n+1]{\frac{2K|\dot{X}(0)|^{n-1}}{(n+1)M}}$$

The generalized Euclidean number Pi in this case is given by

$$\pi_{\phi,\psi} = 2\sin^{-1}_{\phi,\psi}(1) = 2\int_0^1 \frac{ds}{\sqrt{1-s^{n+1}}}$$

Then the frequency is

$$f = \frac{\omega}{2\pi_{\phi,\psi}}$$

When $n = 1$, this solution reduces to the classical solution

$$X(t) = |\dot{X}(0)|\sqrt{\frac{M}{K}}\sin\left(\sqrt{\frac{K}{M}}t\right)$$

for the linear spring-mass equation $M\ddot{X} + KX = 0; X(0) = 0$. It can be observed that for $n \neq 1$, the frequency of the novel generalized harmonic motion of this nonlinear string-mass system $f = \frac{2\pi_{\phi,\psi}}{\omega}$ also depends on the initial velocity of the mass, demonstrating that, even in this simplest model, there is a striking difference between linear and nonlinear mechanical vibration systems. To express the solution in terms of the generalized sine of Drábek and Manásevich and Ôtani, take $p = 2$, and $q = n + 1$, then we have

$$X(t) = A\text{sin}_{2(n+1)}(\omega t)$$

Partial analytic solutions to this problem for $n =$ odd integer can be found in the classical book by S. Timoshenko and D. H. Young and W. Weaver, JR [13]. Our result provides this problem an analytic solution for any legitimate values of n.

Remark: For various trigonometric identities, integrals, and connections to the elementary functions by the David Shelupsky generalized sine function $\sin_p t$, see [14]. For various generalized trigonometric identities and graphic representations of the generalized sine ($\sin_{pq} t$ and $\cos_{pq} t$) of Drábek and Manásevich and M. Ôtani, and the widely cited contributions by [4, 5]. The reader can consult the comprehensive book [11] and their work, e.g., [12]. The paper by P. J. Bushell and D. E. Edmunds [14], provides many useful properties of the $\sin_{pq} t$ function. For important applications, see [15].

Some numerical results are provided here. The solutions for the metals in Table 2 are given in Fig. 3, where $M = 10$. The plots of frequency and amplitude vs. initial velocity $\dot{X}(0)$ respectively for different n are given in Fig. 4.

5 Numerical representations of the generalized sine functions

In this section, numerical solutions of system Eq. (1) are calculated for:

Table 2: K and n for different metals

Metal	K (MPa)	n
Aluminum 1100-O	180	0.20
Aluminum 7075-O	400	0.17
Annealed steel low-carbon	530	0.26
Annealed brass 70-30	895	0.49
Annealed bronze (phosphor)	720	0.46
Annealed copper	315	0.54

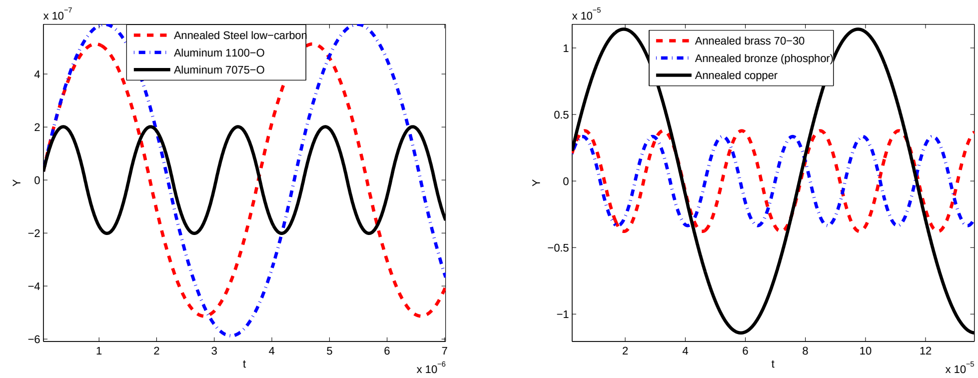


Figure 3: The plot of the $y(t)$ for different metals.

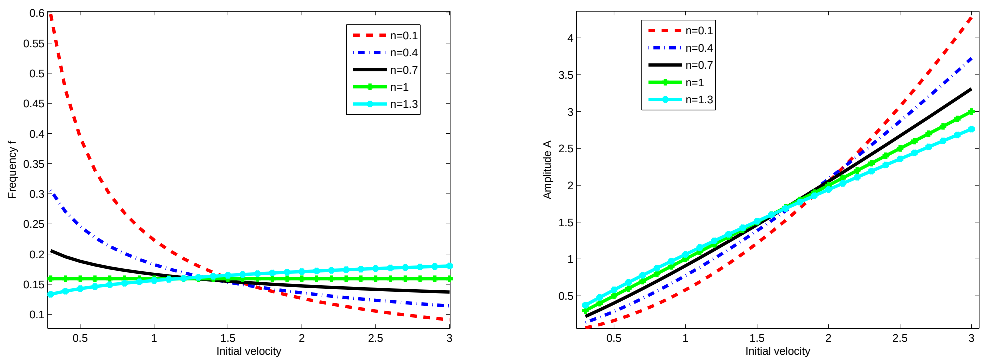


Figure 4: The frequency and amplitude vs. the initial velocity $\dot{y}(0)$, where $K = 1$ and $M = 1$ are assumed.

Case 1, the David Shelupsky's sine and cosine function with

$$\phi(\varepsilon) = \psi(\varepsilon) = |\varepsilon|^{p-2}\varepsilon,\ \ p > 0$$

the results are given in Fig. 5;

Case 2, P. Drábek and R. Manásevich's sine and cosine function with

$$\phi(\varepsilon) = p|\varepsilon|^{p-2}\varepsilon,\ \ \psi(\varepsilon) = q|\varepsilon|^{q-2}\varepsilon,\ \ p, q > 0$$

the results are given in Fig. 6;

Case 3, P. Drábek and R. Manásevich's variation of sine

$$\phi(\varepsilon) = \sqrt[p]{\frac{p-1}{p}}|\varepsilon|^{\frac{2-p}{p-1}}\varepsilon, \psi(\varepsilon) = q|\varepsilon|^{q-2}\varepsilon, \text{and} p > 1, q > 1$$

the results are given in Fig. 7;

Case 4, S. Takeuchi's Jacobi elliptic sine and cosine function with

$$\begin{aligned} \phi(\varepsilon) &= |\varepsilon|^{p-2}\varepsilon,\ \ p > 1 \\ \psi(\varepsilon) &= \frac{q(p-1)}{p}(1 + k^q(1 - 2|\varepsilon|^q)|\varepsilon|^{q-2}\varepsilon,\ \ p > 1, q > 1 \end{aligned}$$

the results are given in Fig. 8. The numerical solutions are obtained by solving numerically the initial value problems for ODE's, which can be readily done by many ODE solvers, such as the Matlab with the function "ode45" based on an explicit Runge-Kutta$(4, 5)$ formula [16].

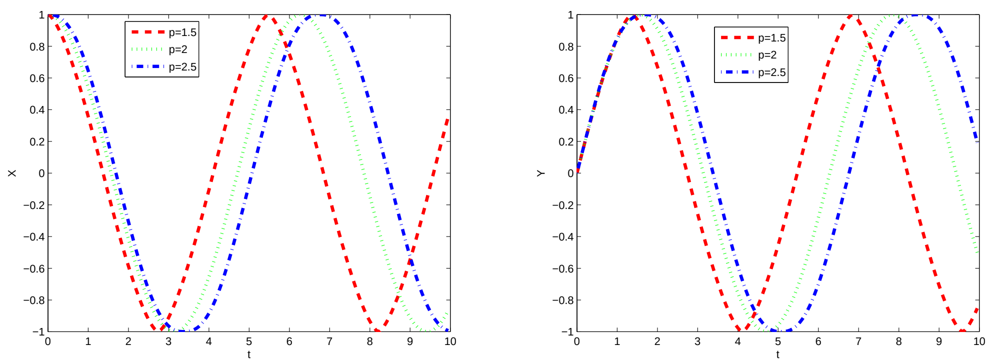


Figure 5: The results of Shelupsky's sine function (Case 1).

6 Conclusions

We have shown the connections between several important generalized sine functions developed by several authors from different perspectives can be

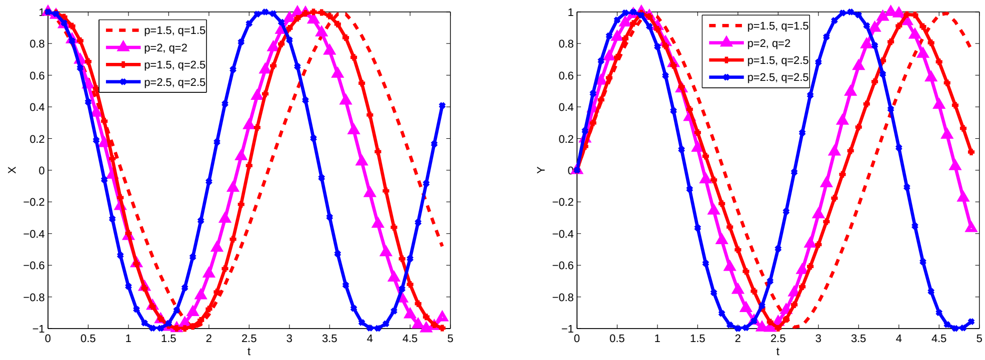


Figure 6: The results of P. Drábek and R. Manásevich's sine function (Case 2).

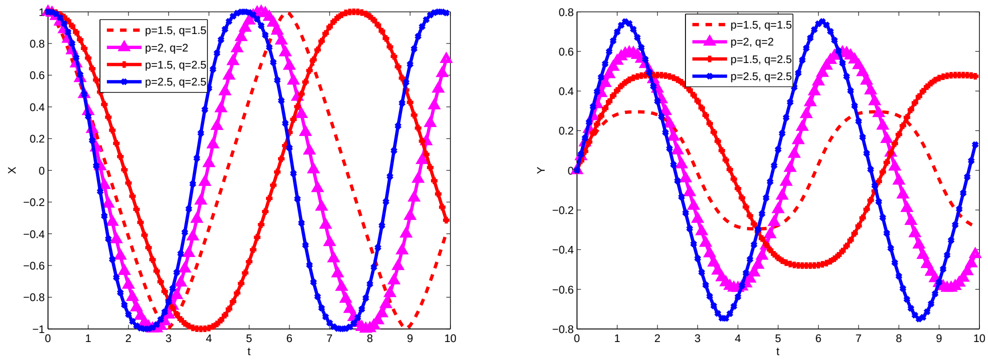


Figure 7: The results of P. Drábek and R. Manásevich's variation of sine function (Case 3).

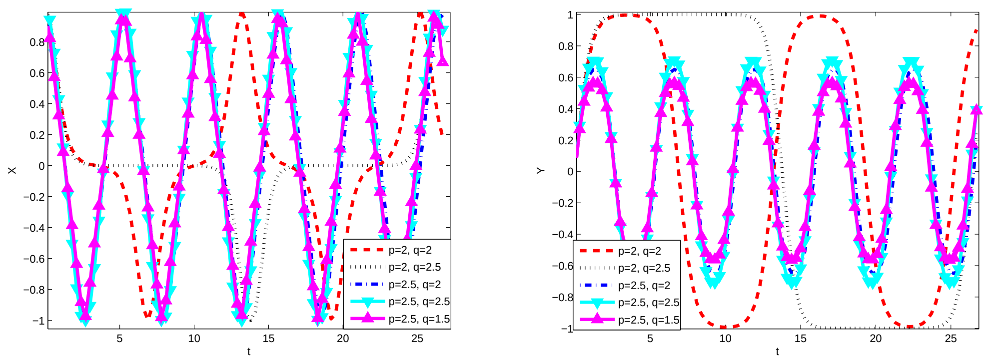


Figure 8: The results of Takeuchi's Jacobi elliptic sine function (Case 4).

casted in the framework of the Hamilton system of first order ODE's in Lagrange Mechanics. This ODE's can be solved by a uniform approach to numerically evaluate the sine functions. We also provide an analytic solution to a nonlinear spring-mass system in terms of one of the generalized sine functions, demonstrating the applicability of these sine functions in solving nonlinear vibration problems in engineering.

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