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Falling Pseudo d-Ideals and d-Subalgebras in Pseudo d-Algebras

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Abstract

Based on the theory of a falling shadow which was first formulated by Wang [15], a theoretical approach of the pseudo ideal structure in pseudo d-algebras is established. Several properties are investigated, and relations among a falling pseudo d-subalgebra, a falling pseudo BCK-ideal and a falling pseudo d-ideal are discussed. A characterization of a falling pseudo d-ideal is provided. Conditions for a falling shadow to be a falling pseudo BCK-ideal, and for a falling pseudo BCK-ideal to be a falling pseudo d-ideal are considered.

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1 Introduction

Imai and Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([3], [4]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. BCK-algebras have several connections with other areas of investigation, such as: lattice ordered groups, MV-algebras, Wajsberg algebras, and implicative commutative semigroups. Font et al. ([1]) have discussed Wajsberg algebras which are term-equivalent to MV-algebras. Neggers and Kim ([10]) introduced the notion of d-algebras which is another useful generalization of BCK-algebras. They investigated several relations between d-algebras and BCK-algebras as well as several other relations between d-algebras and oriented digraphs. After that, some further aspects were studied in [8] and [11]. Neggers et al. ([9]) introduced the concept of d-fuzzy function which generalizes the concept of fuzzy subalgebra to a much larger class of functions in a natural way. In addition, they discussed a method of fuzzification of a wide class of algebraic systems onto $[0, 1]$ along with some consequences. Jun et al. [7] introduced the notion of a pseudo d-algebra as a generalization of the idea of a d-algebra. They considered the notions of pseudo d-subalgebras, pseudo BCK-ideals and pseudo d-ideals of pseudo d-algebras.

In the study of a unified treatment of uncertainty modelled by means of combining probability and fuzzy set theory, Goodman [2] pointed out the equivalence of a fuzzy set and a class of random sets. Wang and Sanchez [14] introduced the theory of falling shadows which directly relates probability concepts with the membership function of fuzzy sets. Falling shadow representation theory shows us the way of selection relaid on the joint degrees distributions. It is reasonable and convenient approach for the theoretical development and the practical applications of fuzzy sets and fuzzy logics. The mathematical structure of the theory of falling shadows is formulated in [15]. Tan et al. [12, 13] established a theoretical approach to define a fuzzy inference relation and fuzzy set operations based on the theory of falling shadows. Jun and Kang [5] established a theoretical approach to define a fuzzy positive implicative ideal in a BCK-algebra based on the theory of falling shadows. They provided relations between falling fuzzy positive implicative ideals and falling fuzzy ideals. They also considered relations between fuzzy positive implicative ideals and falling fuzzy positive implicative ideals. Jun and Kang [6] considered the fuzzification of generalized Tarski filters of generalized Tarski algebras, and investigated related properties. They established characterizations of a fuzzy generalized Tarski filter, and introduced the notion of falling fuzzy generalized Tarski filters in generalized Tarski algebras based on the theory of falling shadows. They provided relations between fuzzy generalized Tarski filters and falling fuzzy generalized Tarski filters, and established a characterization of a falling fuzzy generalized Tarski filter.

In this paper, we establish a theoretical approach to define a falling pseudo d-subalgebra, a falling pseudo d-ideal and a falling pseudo BCK-ideal in pseudo d-algebras based on the theory of falling shadows which was first formulated by Wang [15], and investigate several related properties. We provide relations among a falling pseudo d-subalgebra, a falling pseudo d-ideal and a falling pseudo BCK-ideal. We consider a characterization of a falling pseudo d-ideal. We give conditions for a falling shadow to be a falling pseudo BCK-ideal, and for a falling pseudo BCK-ideal to be a falling pseudo d-ideal.

2 Preliminaries

Based on the papers [7, 11], we display basic definitions and results.

A *d-algebra* is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying the following axioms:

(I) $x * x = 0$,

(II) $0 * x = 0$,

(III) $x * y = 0$ and $y * x = 0$ imply $x = y$

for all x, y in X.

A *pseudo d-algebra* is an algebra $(X; \rightarrow, \rightsquigarrow, 0)$ of type $(2, 2, 0)$ in which the following axioms hold for all $x, y \in X$:

(a1) $x \rightarrow x = x \rightsquigarrow x = 0$,

(a2) $0 \rightarrow x = 0 \rightsquigarrow x = 0$,

(a3) $x \rightarrow y = y \rightsquigarrow x = 0$ implies $x = y$.

Note that if $(X; \rightarrow, 0)$ is a d-algebra, then letting $x \rightsquigarrow y = x \rightarrow y$, produces a pseudo d-algebra $(X; \rightarrow, \rightsquigarrow, 0)$. Hence every d-algebra is a pseudo d-algebra in a natural way.

A nonempty subset S of X is called a *pseudo d-subalgebra* of X if $x \rightarrow y \in S$ and $x \rightsquigarrow y \in S$ whenever $x, y \in S$.

A subset I of X is called a *pseudo BCK-ideal* of X if it satisfies:

(b1) $0 \in I$,

(b2) $(\forall x \in X)\,(\forall y \in I)\,(x \rightarrow y \in I \text{ and } x \rightsquigarrow y \in I \;\Rightarrow\; x \in I)$.

A subset I of X is called a *pseudo d-ideal* of X if it satisfies (b2) and

(b3) $(\forall x, y \in X)\,(x \in I \;\Rightarrow\; x \rightarrow y \in I \text{ and } x \rightsquigarrow y \in I)$.

We now display the basic theory on falling shadows. We refer the reader to the papers [2, 12, 13, 14, 15] for further information regarding the theory of falling shadows.

Given a universe of discourse U, let $\mathrm{P}(U)$ denote the power set of U. For each $u \in U$, let

$$\dot{u} := \{E \mid u \in E \text{ and } E \subseteq U\}, \tag{2.1}$$

and for each $E \in \mathrm{P}(U)$, let

$$\dot{E} := \{\dot{u} \mid u \in E\}. \tag{2.2}$$

An ordered pair $(\mathrm{P}(U), \mathrm{B})$ is said to be a hyper-measurable structure on U if B is a σ-field in $\mathrm{P}(U)$ and $\dot{U} \subseteq \mathrm{B}$. Given a probability space (Ω, A, P) and a hyper-measurable structure $(\mathrm{P}(U), \mathrm{B})$ on U, a random set on U is defined to be a mapping $\xi : \Omega \to \mathrm{P}(U)$ which is A-B measurable, that is,

$$(\forall C \in \mathrm{B})\,(\xi^{-1}(C) = \{\omega \mid \omega \in \Omega \text{ and } \xi(\omega) \in C\} \in \mathrm{A}). \tag{2.3}$$

Suppose that ξ is a random set on U. Let

$$\tilde{H}(u) := P(\omega \mid u \in \xi(\omega)) \text{ for each } u \in U.$$

Then $\tilde{H}$ is a kind of fuzzy set in U. We call $\tilde{H}$ a falling shadow of the random set ξ, and ξ is called a cloud of $\tilde{H}$.

For example, $(\Omega, \mathrm{A}, P) = ([0,1], \mathrm{A}, m)$, where A is a Borel field on $[0,1]$ and m is the usual Lebesgue measure. Let $\tilde{H}$ be a fuzzy set in U and $\tilde{H}_t := \{u \in U \mid \tilde{H}(u) \geq t\}$ be a t-cut of $\tilde{H}$. Then

$$\xi : [0,1] \to \mathrm{P}(U),\ t \mapsto \tilde{H}_t$$

is a random set and ξ is a cloud of $\tilde{H}$. We shall call ξ defined above as the cut-cloud of $\tilde{H}$ (see [2]).

3 Falling pseudo d-subalgebras/ideals

In what follows let X denote a pseudo d-algebra unless otherwise specified.

Definition 3.1. For a probability space (Ω, A, P) and a random set $\xi : \Omega \to \mathrm{P}(X)$, the falling shadow $\tilde{H}$ of ξ is called a *falling pseudo d-subalgebra* (resp. *falling pseudo BCK-ideal* and *falling pseudo d-ideal*) of X if $\xi(\omega)$ is a pseudo d-subalgebra (resp. pseudo BCK-ideal and pseudo d-ideal) of X for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$.

Obviously, every falling pseudo d-ideal is a falling pseudo d-subalgebra, but the converse is not true as seen in the following example.

Example 3.2. Let $X := \{0, a, b, c\}$ be a set with the following two Cayley tables:

$\rightarrow$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	b	0	b
c	c	0	c	0

$\rightsquigarrow$	0	a	b	c
0	0	0	0	0
a	a	0	0	c
b	b	b	0	0
c	c	c	0	0

Then $(X; \rightarrow, 0)$ and $(X; \rightsquigarrow, 0)$ are not d-algebras, but $(X; \rightarrow, \rightsquigarrow, 0)$ is a pseudo d-algebra (see [7]). For a probability space $(\Omega, \mathrm{A}, P) = ([0, 1], \mathrm{A}, m)$, define a random set

$$\xi : \Omega \rightarrow \mathrm{P}(X), \quad \omega \mapsto \begin{cases} \{0, a\} & \text{if } t \in [0, 0.6), \\ X & \text{if } t \in [0.6, 0.8), \\ \emptyset & \text{if } t \in [0.8, 1]. \end{cases}$$

Then the falling shadow $\tilde{H}$ of ξ is a falling pseudo d-subalgebra of X. If we take $t \in [0, 0.6)$, then $\xi(t) = \{0, a\}$ is not a pseudo d-ideal of X. Hence $\tilde{H}$ is not a falling pseudo d-ideal of X.

We provide an example of a falling pseudo d-ideal which is not a falling pseudo BCK-ideal.

Example 3.3. Let $X := \{0, a, b, c\}$ be a set with the following two Cayley tables:

$\rightarrow$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	0	0	b
c	c	c	a	0

$\rightsquigarrow$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	0
c	c	c	0	0

Then $(X; \rightarrow, 0)$ and $(X; \rightsquigarrow, 0)$ are not d-algebras, but $(X; \rightarrow, \rightsquigarrow, 0)$ is a pseudo d-algebra (see [7]). For a probability space $(\Omega, \mathrm{A}, P) = ([0, 1], \mathrm{A}, m)$, define a random set

$$\xi : \Omega \rightarrow \mathrm{P}(X), \quad \omega \mapsto \begin{cases} \{0, a\} & \text{if } t \in [0, 0.7), \\ \emptyset & \text{if } t \in [0.7, 0.8), \\ X & \text{if } t \in [0.8, 1]. \end{cases}$$

Then the falling shadow $\tilde{H}$ of ξ is a falling pseudo d-ideal of X, but not a falling pseudo BCK-ideal of X.

The following example shows that any falling pseudo BCK-ideal is neither a falling pseudo d-subalgebra nor a falling pseudo d-ideal.

Example 3.4. Let $X := \{0, a, b, c\}$ be a set with the following two Cayley tables:

$\rightarrow$	0	a	b	c
0	0	0	0	0
a	0	0	a	a
b	b	0	0	b
c	c	c	b	0

$\rightsquigarrow$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	0	0	a
c	c	c	c	0

Then $(X; \rightarrow, 0)$ and $(X; \rightsquigarrow, 0)$ are not d-algebras, but $(X; \rightarrow, \rightsquigarrow, 0)$ is a pseudo d-algebra. For a probability space $(\Omega, \mathrm{A}, P) = ([0, 1], \mathrm{A}, m)$, define a random set

$$\xi : \Omega \rightarrow \mathrm{P}(X), \quad \omega \mapsto \begin{cases} \{0, b, c\} & \text{if } t \in [0, 0.7), \\ X & \text{if } t \in [0.7, 0.9), \\ \emptyset & \text{if } t \in [0.9, 1]. \end{cases}$$

Then the falling shadow $\tilde{H}$ of ξ is a falling pseudo BCK-ideal of X. If we take $t \in [0, 0.7)$, then $\xi(t) = \{0, b, c\}$ is not a pseudo d-subalgebra of X since $b \rightarrow c = b \in \{0, b, c\}$ and $b \rightsquigarrow c = a \notin \{0, b, c\}$. Hence $\tilde{H}$ is not a falling pseudo d-subalgebra of X. Also $\xi(t)$ is not a pseudo d-ideal of X, since $b \rightarrow c = b \in \{0, b, c\}$ but $b \rightsquigarrow c = a \notin \{0, b, c\}$. Hence $\tilde{H}$ is not a falling pseudo d-ideal of X.

The following example shows that any falling pseudo d-subalgebra is not a falling pseudo BCK-ideal.

Example 3.5. Consider a pseudo d-algebra $X := \{0, a, b, c\}$ as in Example 3.3. For a probability space $(\Omega, \mathrm{A}, P) = ([0, 1], \mathrm{A}, m)$, define a random set

$$\xi : \Omega \rightarrow \mathrm{P}(X), \quad \omega \mapsto \begin{cases} \{0, a, b\} & \text{if } t \in [0, 0.5), \\ \emptyset & \text{if } t \in [0.5, 0.8), \\ X & \text{if } t \in [0.8, 1]. \end{cases}$$

Then the falling shadow $\tilde{H}$ of ξ is a falling pseudo d-subalgebra of X. If we take $t \in [0, 0.5)$, then $\xi(t) = \{0, a, b\}$ is not a pseudo BCK-ideal of X, since $c \rightarrow b = a, c \rightsquigarrow b = 0 \in \{0, a, b\}$, but $c \notin\in \{0, a, b\}$. Hence $\tilde{H}$ is not a falling pseudo BCK-ideal of X.

Definition 3.6. If a pseudo d-algebra X satisfies the following condition:

$$(\forall x, y \in X)\,((x \to y) \rightsquigarrow x = 0 \text{ and } (x \rightsquigarrow y) \to x = 0)\,,$$

then we say X is a *pseudo d^*-algebra.*

Example 3.7. Let $X := \{0, a, b, c\}$ be a set with the following two Cayley tables:

$\to$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	0	0	b
c	c	c	a	0

$\rightsquigarrow$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	0	0

Then $(X; \to, 0)$ and $(X; \rightsquigarrow, 0)$ are not d-algebras, but $(X; \to, \rightsquigarrow, 0)$ is a pseudo d-algebra which is also a pseudo d^*-algebra.

Theorem 3.8. *In a pseudo d^*-algebra, every falling pseudo BCK-ideal is a falling pseudo d-ideal.*

Proof. Let $\tilde{H}$ be a falling pseudo BCK-ideal of a pseudo d^*-algebra X. Then $\xi(\omega)$ is a pseudo BCK-ideal of X for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$. Let $x, y \in X$ and $\omega \in \Omega$ be such that $x \in \xi(\omega)$. Since X is a d^*-algebra, we have $(x \to y) \rightsquigarrow x = 0 \in \xi(\omega)$ and $(x \rightsquigarrow y) \to x = 0 \in \xi(\omega)$ for all $x, y \in X$. Using (b2), we get $x \to y \in \xi(\omega)$ and $x \rightsquigarrow y \in \xi(\omega)$ for all $y \in X$. Hence $\xi(\omega)$ is a pseudo d-ideal of X, and thus $\tilde{H}$ is a falling pseudo d-ideal of X. □

Corollary 3.9. *In a pseudo d^*-algebra, every falling pseudo BCK-ideal is a falling pseudo d-subalgebra.*

The following example shows that there exists a falling pseudo d-subalgebra which is not a falling pseudo BCK-ideal in pseudo d^*-algebras.

Example 3.10. Consider a pseudo d^*-algebra $X = \{0, a, b, c\}$ as in Example 3.7. For a probability space $(\Omega, \mathrm{A}, P) = ([0, 1], \mathrm{A}, m)$, define a random set

$$\xi : \Omega \to \mathrm{P}(X), \quad \omega \mapsto \begin{cases} \{0, a, b\} & \text{if } t \in [0, 0.7), \\ X & \text{if } t \in [0.7, 0.8), \\ \emptyset & \text{if } t \in [0.8, 1]. \end{cases}$$

Then the falling shadow $\tilde{H}$ of ξ is a falling pseudo d-subalgebra of X. If we take $t \in [0, 0.7)$, then $\xi(t) = \{0, a, b\}$ is not a pseudo BCK-ideal of X since $c \to b = a, c \rightsquigarrow b = 0 \in \{0, a, b\}$ and $c \notin \{0, a, b\}$. Hence $\tilde{H}$ is not a falling pseudo BCK-ideal of X.

Let (Ω, A, P) be a probability space and $\tilde{H}$ a falling shadow of a random set $\xi : \Omega \to \mathrm{P}(X)$. For any $x \in X$, let

$$\Omega(x;\xi) := \{\omega \in \Omega \mid x \in \xi(\omega)\}.$$

Then $\Omega(x;\xi) \in \mathrm{A}$.

Lemma 3.11. *If $\tilde{H}$ is a falling pseudo d-subalgebra of X then*

$$(\forall x \in X)\,(\Omega(x;\xi) \subseteq \Omega(0;\xi))\,. \tag{3.1}$$

Proof. If $\Omega(x;\xi) = \emptyset$, then it is clear. Assume that $\Omega(x;\xi) \neq \emptyset$ and let $\omega \in \Omega$ be such that $\omega \in \Omega(x;\xi)$. Then $x \in \xi(\omega)$, and so $0 = x \to x = x \rightsquigarrow x \in \xi(\omega)$ since $\xi(\omega)$ is a pseudo d-subalgebra of X. Hence $\omega \in \Omega(0;\xi)$, and thus $\Omega(x;\xi) \subseteq \Omega(0;\xi)$ for all $x \in X$. □

Corollary 3.12. *If $\tilde{H}$ is a falling pseudo d-ideal of X, then* (3.1) *is valid.*

Theorem 3.13. *Let $\tilde{H}$ be a falling shadow of a random set ξ on X. Then $\tilde{H}$ is a falling pseudo d-ideal of X if and only if the following conditions are valid:*

(a) $(\forall x, y \in X)\,(\Omega(x \to y;\xi) \cap \Omega(y;\xi) \cap \Omega(x \rightsquigarrow y;\xi) \subseteq \Omega(x;\xi))\,,$

(b) $(\forall x, y \in X)\,(\Omega(x;\xi) \subseteq \Omega(x \to y;\xi) \cap \Omega(x \rightsquigarrow y;\xi))\,.$

Proof. Suppose that $\tilde{H}$ is a falling pseudo d-ideal of X. Let $x, y \in X$ and $\omega \in \Omega$. If $\omega \in \Omega(x \to y;\xi) \cap \Omega(y;\xi) \cap \Omega(x \rightsquigarrow y;\xi)$, then $y \in \xi(\omega)$, $x \to y \in \xi(\omega)$ and $x \rightsquigarrow y \in \xi(\omega)$. Since $\xi(\omega)$ is a pseudo d-ideal of X, it follows from (b2) that $x \in \xi(\omega)$ so that $\omega \in \Omega(x;\xi)$. Hence

$$\Omega(x \to y;\xi) \cap \Omega(y;\xi) \cap \Omega(x \rightsquigarrow y;\xi) \subseteq \Omega(x;\xi).$$

If $\omega \in \Omega(x;\xi)$, then $x \in \xi(\omega)$ and so $x \to y \in \xi(\omega)$ and $x \rightsquigarrow y \in \xi(\omega)$ by (b3). Hence $\omega \in \Omega(x \to y;\xi) \cap \Omega(x \rightsquigarrow y;\xi)$, and thus $\Omega(x;\xi) \subseteq \Omega(x \to y;\xi) \cap \Omega(x \rightsquigarrow y;\xi)$.

Conversely, suppose that two conditions (a) and (b) are valid. Let $x, y \in X$ and $\omega \in \Omega$ be such that $x \to y \in \xi(\omega)$, $x \rightsquigarrow y \in \xi(\omega)$ and $y \in \xi(\omega)$. Then $\omega \in \Omega(x \to y;\xi)$, $\omega \in \Omega(x \rightsquigarrow y;\xi)$ and $\omega \in \Omega(y;\xi)$. It follows from (a) that

$$\omega \in \Omega(x \to y;\xi) \cap \Omega(y;\xi) \cap \Omega(x \rightsquigarrow y;\xi) \subseteq \Omega(x;\xi)$$

so that $x \in \xi(\omega)$. Now assume that $x \in \xi(\omega)$ for every $x \in X$ and $\omega \in \Omega$. Then

$$\omega \in \Omega(x;\xi) \subseteq \Omega(x \to y;\xi) \cap \Omega(x \rightsquigarrow y;\xi)$$

for all $y \in X$ by (b). Hence $x \to y \in \xi(\omega)$ and $x \rightsquigarrow y \in \xi(\omega)$ for all $y \in X$. Therefore $\tilde{H}$ is a falling pseudo d-ideal of X. □

Corollary 3.14. *For a falling shadow $\tilde{H}$ of a random set ξ on X, if the conditions* (a) *and* (b) *in Theorem 3.13 hold, then $\tilde{H}$ is a falling pseudo d-subalgebra of X.*

Proposition 3.15. *For a falling shadow $\tilde{H}$ of a random set ξ on X, if $\tilde{H}$ is a falling pseudo BCK-ideal or a falling pseudo d-ideal of X, then*

$$(\forall x, y \in X)\,(x \to y = 0 = x \rightsquigarrow y \;\Rightarrow\; \Omega(y;\xi) \subseteq \Omega(x;\xi))\,. \tag{3.2}$$

Proof. Assume that $\tilde{H}$ is a falling pseudo d-ideal of X. Let $x, y \in X$ and $\omega \in \Omega$ be such that $x \to y = 0 = x \rightsquigarrow y$ and $\omega \in \Omega(y;\xi)$. Then $y \in \xi(\omega)$ and $\omega \in \Omega(0;\xi)$ by Corollary 3.12. Hence $x \to y = 0 \in \xi(\omega)$ and $x \rightsquigarrow y = 0 \in \xi(\omega)$. Since $\xi(\omega)$ is a pseudo d-ideal of X, it follows from (b2) that $x \in \xi(\omega)$ so that $\omega \in \Omega(x;\xi)$. Therefore $\Omega(y;\xi) \subseteq \Omega(x;\xi)$ for all $x, y \in X$ with $x \to y = 0 = x \rightsquigarrow y$.

Now suppose that $\tilde{H}$ is a falling pseudo BCK-ideal of X. Let $x, y \in X$ and $\omega \in \Omega$ be such that $x \to y = 0 = x \rightsquigarrow y$ and $\omega \in \Omega(y;\xi)$. Then $y \in \xi(\omega)$ and $x \to y = x \rightsquigarrow y = 0 \in \xi(\omega)$ by (b1). It follows from (b2) that $x \in \xi(\omega)$ so that $\omega \in \Omega(x;\xi)$. Hence $\Omega(y;\xi) \subseteq \Omega(x;\xi)$ for all $x, y \in X$ with $x \to y = 0 = x \rightsquigarrow y$. □

Proposition 3.16. *For a falling shadow $\tilde{H}$ of a random set ξ on X, if $\tilde{H}$ is a falling pseudo BCK-ideal or a falling pseudo d-ideal of X, then*

$$(\forall x, y \in X)\,(\Omega(x \to y;\xi) \cap \Omega(y;\xi) \cap \Omega(x \rightsquigarrow y;\xi) \subseteq \Omega(x;\xi))\,, \tag{3.3}$$

Proof. Let $x, y \in X$ and $\omega \in \Omega$ be such that $\omega \in \Omega(x \to y;\xi) \cap \Omega(y;\xi) \cap \Omega(x \rightsquigarrow y;\xi)$. Then $x \to y \in \xi(\omega)$, $y \in \xi(\omega)$ and $x \rightsquigarrow y \in \xi(\omega)$. Since $\xi(\omega)$ is a pseudo BCK-ideal or a pseudo d-ideal of X, it follows from (b2) that $x \in \xi(\omega)$ so that $\omega \in \Omega(x;\xi)$. Hence

$$\Omega(x \to y;\xi) \cap \Omega(y;\xi) \cap \Omega(x \rightsquigarrow y;\xi) \subseteq \Omega(x;\xi)$$

for all $x, y \in X$. □

We give conditions for a falling shadow to be a falling pseudo BCK-ideal.

Theorem 3.17. *For a falling shadow $\tilde{H}$ of a random set ξ on X, assume that the following conditions are satisfied:*

(a) $\Omega = \Omega(0;\xi)$,

(b) $(\forall x, y \in X)\,(\Omega(x \to y;\xi) \cap \Omega(y;\xi) \cap \Omega(x \rightsquigarrow y;\xi) \subseteq \Omega(x;\xi))\,.$

Then $\tilde{H}$ is a falling pseudo BCK-ideal of X.

Proof. Using (a), we have $0 \in \xi(\omega)$ for all $\omega \in \Omega$. Let $x, y \in X$ and $\omega \in \Omega$ be such that $x \to y \in \xi(\omega)$, $y \in \xi(\omega)$ and $x \rightsquigarrow y \in \xi(\omega)$. Then

$$\omega \in \Omega(x \to y; \xi) \cap \Omega(y; \xi) \cap \Omega(x \rightsquigarrow y; \xi) \subseteq \Omega(x; \xi)$$

by (b), and so $x \in \xi(\omega)$. Therefore $\xi(\omega)$ is a pseudo BCK-ideal of X for all $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$. Hence $\tilde{H}$ is a falling pseudo BCK-ideal of X. □

Corollary 3.18. *For a falling shadow $\tilde{H}$ of a random set ξ on X, assume that two conditions* (a) *and* (b) *in Theorem 3.17 are satisfied. If X is a pseudo d^*-algebra, then $\tilde{H}$ is a falling pseudo d-ideal of X.*

Theorem 3.19. *If $\tilde{H}$ is a falling pseudo d-subalgebra of X, then*

$$(\forall x, y \in X) \left(\min\{\tilde{H}(x \to y), \tilde{H}(x \rightsquigarrow y)\} \geq T_m(\tilde{H}(x), \tilde{H}(y)) \right) \tag{3.4}$$

where $T_m(s,t) = \max\{s + t - 1, 0\}$ for any $s, t \in [0, 1]$.

Proof. By Definition 3.1, $\xi(\omega)$ is a pseudo d-subalgebra of X for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$. Hence

$$\{\omega \in \Omega \mid x \in \xi(\omega)\} \cap \{\omega \in \Omega \mid y \in \xi(\omega)\} \subseteq \{\omega \in \Omega \mid x \to y \in \xi(\omega)\}$$

and

$$\{\omega \in \Omega \mid x \in \xi(\omega)\} \cap \{\omega \in \Omega \mid y \in \xi(\omega)\} \subseteq \{\omega \in \Omega \mid x \rightsquigarrow y \in \xi(\omega)\}$$

which imply that

$$\begin{aligned} \tilde{H}(x \to y) &= P(\omega \mid x \to y \in \xi(\omega)) \\ &\geq P(\{\omega \mid x \in \xi(\omega)\} \cap \{\omega \mid y \in \xi(\omega)\}) \\ &\geq P(\omega \mid x \in \xi(\omega)) + P(\omega \mid y \in \xi(\omega)) \\ &\qquad - P(\omega \mid x \in \xi(\omega) \text{ or } y \in \xi(\omega)) \\ &\geq \tilde{H}(x) + \tilde{H}(y) - 1 \end{aligned}$$

and

$$\begin{aligned} \tilde{H}(x \rightsquigarrow y) &= P(\omega \mid x \rightsquigarrow y \in \xi(\omega)) \\ &\geq P(\{\omega \mid x \in \xi(\omega)\} \cap \{\omega \mid y \in \xi(\omega)\}) \\ &\geq P(\omega \mid x \in \xi(\omega)) + P(\omega \mid y \in \xi(\omega)) \\ &\qquad - P(\omega \mid x \in \xi(\omega) \text{ or } y \in \xi(\omega)) \\ &\geq \tilde{H}(x) + \tilde{H}(y) - 1. \end{aligned}$$

Hence

$$\tilde{H}(x \to y) \geq \max\{\tilde{H}(x) + \tilde{H}(y) - 1, 0\} = T_m(\tilde{H}(x), \tilde{H}(y))$$

and

$$\tilde{H}(x \rightsquigarrow y) \geq \max\{\tilde{H}(x) + \tilde{H}(y) - 1, 0\} = T_m(\tilde{H}(x), \tilde{H}(y)).$$

Therefore $\min\{\tilde{H}(x \to y), \tilde{H}(x \rightsquigarrow y)\} \geq T_m(\tilde{H}(x), \tilde{H}(y))$ for all $x, y \in X$. □

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