Falling Pseudo $d$-Ideals and $d$-Subalgebras
in Pseudo $d$-Algebras

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Abstract

Based on the theory of a falling shadow which was first formulated by
Wang [15], a theoretical approach of the pseudo ideal structure in pseudo
d-algebras is established. Several properties are investigated, and rela-
tions among a falling pseudo $d$-subalgebra, a falling pseudo $BCK$-ideal
and a falling pseudo $d$-ideal are discussed. A characterization of a falling
pseudo $d$-ideal is provided. Conditions for a falling shadow to be a falling
pseudo $BCK$-ideal, and for a falling pseudo $BCK$-ideal to be a falling
pseudo $d$-ideal are considered.

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pseudo $d$-ideal, (Falling) pseudo $BCK$-ideal.

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1 Introduction

Imai and Iséki introduced two classes of abstract algebras: $BCK$-algebras and $BCI$-algebras ([3], [4]). It is known that the class of $BCK$-algebras is a proper subclass of the class of $BCI$-algebras. $BCK$-algebras have several connections with other areas of investigation, such as: lattice ordered groups, $MV$-algebras, Wajsberg algebras, and implicative commutative semigroups. Font et al. ([1]) have discussed Wajsberg algebras which are term-equivalent to $MV$-algebras. Neggers and Kim ([10]) introduced the notion of $d$-algebras which is another useful generalization of $BCK$-algebras. They investigated several relations between $d$-algebras and $BCK$-algebras as well as several other relations between $d$-algebras and oriented digraphs. After that, some further aspects were studied in [8] and [11]. Neggers et al. ([9]) introduced the concept of $d$-fuzzy function which generalizes the concept of fuzzy subalgebra to a much larger class of functions in a natural way. In addition, they discussed a method of fuzzification of a wide class of algebraic systems onto $[0, 1]$ along with some consequences. Jun et al. [7] introduced the notion of a pseudo $d$-algebra as a generalization of the idea of a $d$-algebra. They considered the notions of pseudo $d$-subalgebras, pseudo $BCK$-ideals and pseudo $d$-ideals of pseudo $d$-algebras.

In this paper, we establish a theoretical approach to define a falling pseudo $d$-subalgebra, a falling pseudo $d$-ideal and a falling pseudo $BCK$-ideal in pseudo $d$-algebras based on the theory of falling shadows which was first formulated by Wang [15], and investigate several related properties. We provide relations among a falling pseudo $d$-subalgebra, a falling pseudo $d$-ideal and a falling pseudo $BCK$-ideal. We consider a characterization of a falling pseudo $d$-ideal. We give conditions for a falling shadow to be a falling pseudo $BCK$-ideal, and for a falling pseudo $BCK$-ideal to be a falling pseudo $d$-ideal.

2 Preliminaries

Based on the papers [7, 11], we display basic definitions and results.

A $d$-algebra is a non-empty set $X$ with a constant $0$ and a binary operation “$\ast$” satisfying the following axioms:

(I) $x \ast x = 0,$

(II) $0 \ast x = 0,$

(III) $x \ast y = 0$ and $y \ast x = 0$ imply $x = y$

for all $x, y$ in $X$.

A pseudo $d$-algebra is an algebra $(X; \rightarrow; \leadsto; 0)$ of type $(2, 2, 0)$ in which the following axioms hold for all $x, y \in X$:

(a1) $x \rightarrow x = x \leadsto x = 0$,

(a2) $0 \rightarrow x = 0 \leadsto x = 0$,

(a3) $x \rightarrow y = y \leadsto x = 0$ implies $x = y$.

Note that if $(X; \rightarrow, 0)$ is a $d$-algebra, then letting $x \leadsto y = x \rightarrow y$, produces a pseudo $d$-algebra $(X; \rightarrow, \leadsto, 0)$. Hence every $d$-algebra is a pseudo $d$-algebra in a natural way.

A nonempty subset $S$ of $X$ is called a pseudo $d$-subalgebra of $X$ if $x \rightarrow y \in S$ and $x \leadsto y \in S$ whenever $x, y \in S$.

A subset $I$ of $X$ is called a pseudo $BCK$-ideal of $X$ if it satisfies:

(b1) $0 \in I,$

(b2) $(\forall x \in X) (\forall y \in I) (x \rightarrow y \in I$ and $x \leadsto y \in I \Rightarrow x \in I).$

A subset $I$ of $X$ is called a pseudo $d$-ideal of $X$ if it satisfies (b2) and

(b3) $(\forall x, y \in X) (x \in I \Rightarrow x \rightarrow y \in I$ and $x \leadsto y \in I).$
We now display the basic theory on falling shadows. We refer the reader to the papers [2, 12, 13, 14, 15] for further information regarding the theory of falling shadows.

Given a universe of discourse $U$, let $P(U)$ denote the power set of $U$. For each $u \in U$, let

$$\dot{u} := \{E \mid u \in E \text{ and } E \subseteq U\}, \quad (2.1)$$

and for each $E \in P(U)$, let

$$\dot{E} := \{\dot{u} \mid u \in E\}. \quad (2.2)$$

An ordered pair $(P(U), B)$ is said to be a hyper-measurable structure on $U$ if $B$ is a $\sigma$-field in $P(U)$ and $\dot{U} \subseteq B$. Given a probability space $(\Omega, A, P)$ and a hyper-measurable structure $(P(U), B)$ on $U$, a random set on $U$ is defined to be a mapping $\xi : \Omega \rightarrow P(U)$ which is $A$-$B$ measurable, that is,

$$(\forall C \in B) (\xi^{-1}(C) = \{\omega \mid \omega \in \Omega \text{ and } \xi(\omega) \in C\} \in A). \quad (2.3)$$

Suppose that $\xi$ is a random set on $U$. Let

$$\tilde{H}(u) := P(\omega \mid u \in \xi(\omega)) \text{ for each } u \in U.$$

Then $\tilde{H}$ is a kind of fuzzy set in $U$. We call $\tilde{H}$ a falling shadow of the random set $\xi$, and $\xi$ is called a cloud of $\tilde{H}$.

For example, $(\Omega, A, P) = ([0, 1], A, m)$, where $A$ is a Borel field on $[0, 1]$ and $m$ is the usual Lebesgue measure. Let $\tilde{H}$ be a fuzzy set in $U$ and $\tilde{H}_t := \{u \in U \mid \tilde{H}(u) \geq t\}$ be a $t$-cut of $\tilde{H}$. Then

$$\xi : [0, 1] \rightarrow P(U), \ t \mapsto \tilde{H}_t$$

is a random set and $\xi$ is a cloud of $\tilde{H}$. We shall call $\xi$ defined above as the cut-cloud of $\tilde{H}$ (see [2]).

### 3 Falling pseudo $d$-subalgebras/ideals

In what follows let $X$ denote a pseudo $d$-algebra unless otherwise specified.

**Definition 3.1.** For a probability space $(\Omega, A, P)$ and a random set $\xi : \Omega \rightarrow P(X)$, the falling shadow $\tilde{H}$ of $\xi$ is called a **falling pseudo $d$-subalgebra** (resp. **falling pseudo $BCK$-ideal** and **falling pseudo $d$-ideal**) of $X$ if $\xi(\omega)$ is a pseudo $d$-subalgebra (resp. pseudo $BCK$-ideal and pseudo $d$-ideal) of $X$ for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$.

Obviously, every falling pseudo $d$-ideal is a falling pseudo $d$-subalgebra, but the converse is not true as seen in the following example.
Example 3.2. Let $X := \{0, a, b, c\}$ be a set with the following two Cayley tables:

\[
\begin{array}{cccc}
\rightarrow & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & 0 & 0 \\
b & b & b & 0 & b \\
c & c & c & a & 0 \\
\end{array}
\quad \quad
\begin{array}{cccc}
\leadsto & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & 0 & c \\
b & b & b & 0 & 0 \\
c & c & c & 0 & 0 \\
\end{array}
\]

Then $(X; \rightarrow, 0)$ and $(X; \leadsto, 0)$ are not $d$-algebras, but $(X; \rightarrow, \leadsto, 0)$ is a pseudo $d$-algebra (see [7]). For a probability space $(\Omega, A, P) = ([0, 1], A, m)$, define a random set

\[
\xi : \Omega \to \mathcal{P}(X), \quad \omega \mapsto \begin{cases} 
\{0, a\} & \text{if } t \in [0, 0.6), \\
X & \text{if } t \in [0.6, 0.8), \\
\emptyset & \text{if } t \in [0.8, 1]. 
\end{cases}
\]

Then the falling shadow $\tilde{H}$ of $\xi$ is a falling pseudo $d$-subalgebra of $X$. If we take $t \in [0, 0.6)$, then $\xi(t) = \{0, a\}$ is not a pseudo $d$-ideal of $X$. Hence $\tilde{H}$ is not a falling pseudo $d$-ideal of $X$.

We provide an example of a falling pseudo $d$-ideal which is not a falling pseudo $BCK$-ideal.

Example 3.3. Let $X := \{0, a, b, c\}$ be a set with the following two Cayley tables:

\[
\begin{array}{cccc}
\rightarrow & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & 0 & a \\
b & b & 0 & 0 & b \\
c & c & c & a & 0 \\
\end{array}
\quad \quad
\begin{array}{cccc}
\leadsto & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & a & a \\
b & b & b & 0 & 0 \\
c & c & c & 0 & 0 \\
\end{array}
\]

Then $(X; \rightarrow, 0)$ and $(X; \leadsto, 0)$ are not $d$-algebras, but $(X; \rightarrow, \leadsto, 0)$ is a pseudo $d$-algebra (see [7]). For a probability space $(\Omega, A, P) = ([0, 1], A, m)$, define a random set

\[
\xi : \Omega \to \mathcal{P}(X), \quad \omega \mapsto \begin{cases} 
\{0, a\} & \text{if } t \in [0, 0.7), \\
\emptyset & \text{if } t \in [0.7, 0.8), \\
X & \text{if } t \in [0.8, 1]. 
\end{cases}
\]

Then the falling shadow $\tilde{H}$ of $\xi$ is a falling pseudo $d$-ideal of $X$, but not a falling pseudo $BCK$-ideal of $X$. 
The following example shows that any falling pseudo $BCK$-ideal is neither a falling pseudo $d$-subalgebra nor a falling pseudo $d$-ideal.

**Example 3.4.** Let $X := \{0, a, b, c\}$ be a set with the following two Cayley tables:

<table>
<thead>
<tr>
<th>$\rightarrow$</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>a</td>
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<td>a</td>
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<td>b</td>
<td>b</td>
<td>0</td>
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<td>b</td>
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<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>b</td>
<td>0</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>$\Rightarrow$</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>a</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

Then $(X; \rightarrow, 0)$ and $(X; \Rightarrow, 0)$ are not $d$-algebras, but $(X; \rightarrow, \Rightarrow, 0)$ is a pseudo $d$-algebra. For a probability space $(\Omega, A, P) = ([0, 1], A, m)$, define a random set

$$\xi : \Omega \rightarrow P(X), \; \omega \mapsto \begin{cases} \{0, b, c\} & \text{if } t \in [0, 0.7), \\ X & \text{if } t \in [0.7, 0.9), \\ \emptyset & \text{if } t \in [0.9, 1]. \end{cases}$$

Then the falling shadow $\tilde{H}$ of $\xi$ is a falling pseudo $BCK$-ideal of $X$. If we take $t \in [0, 0.7)$, then $\xi(t) = \{0, b, c\}$ is not a pseudo $d$-subalgebra of $X$ since $b \rightarrow c = b \in \{0, b, c\}$ and $b \Rightarrow c = a \notin \{0, b, c\}$. Hence $\tilde{H}$ is not a falling pseudo $d$-subalgebra of $X$. Also $\xi(t)$ is not a pseudo $d$-ideal of $X$, since $b \rightarrow c = b \in \{0, b, c\}$ but $b \Rightarrow c = a \notin \{0, b, c\}$. Hence $\tilde{H}$ is not a falling pseudo $d$-ideal of $X$.

The following example shows that any falling pseudo $d$-subalgebra is not a falling pseudo $BCK$-ideal.

**Example 3.5.** Consider a pseudo $d$-algebra $X := \{0, a, b, c\}$ as in Example 3.3. For a probability space $(\Omega, A, P) = ([0, 1], A, m)$, define a random set

$$\xi : \Omega \rightarrow P(X), \; \omega \mapsto \begin{cases} \{0, a, b\} & \text{if } t \in [0, 0.5), \\ \emptyset & \text{if } t \in [0.5, 0.8), \\ X & \text{if } t \in [0.8, 1]. \end{cases}$$

Then the falling shadow $\tilde{H}$ of $\xi$ is a falling pseudo $d$-subalgebra of $X$. If we take $t \in [0, 0.5)$, then $\xi(t) = \{0, a, b\}$ is not a pseudo $BCK$-ideal of $X$, since $c \rightarrow b = a, c \Rightarrow b = 0 \in \{0, a, b\}$, but $c \notin \{0, a, b\}$. Hence $\tilde{H}$ is not a falling pseudo $BCK$-ideal of $X$. 


Definition 3.6. If a pseudo $d$-algebra $X$ satisfies the following condition:
\[(\forall x, y \in X) ((x \rightarrow y) \leadsto x = 0 \text{ and } (x \leadsto y) \rightarrow x = 0),\]
then we say $X$ is a pseudo $d^*$-algebra.

Example 3.7. Let $X := \{0, a, b, c\}$ be a set with the following two Cayley tables:

<table>
<thead>
<tr>
<th>$\rightarrow$</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>a</td>
<td>a</td>
<td>0</td>
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<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>0</td>
<td>0</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\leadsto$</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>a</td>
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<tr>
<td>b</td>
<td>b</td>
<td>b</td>
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<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Then $(X; \rightarrow, 0)$ and $(X; \leadsto, 0)$ are not $d$-algebras, but $(X; \rightarrow, \leadsto, 0)$ is a pseudo $d$-algebra which is also a pseudo $d^*$-algebra.

Theorem 3.8. In a pseudo $d^*$-algebra, every falling pseudo $BCK$-ideal is a falling pseudo $d$-ideal.

Proof. Let $\tilde{H}$ be a falling pseudo $BCK$-ideal of a pseudo $d^*$-algebra $X$. Then $\xi(\omega)$ is a pseudo $BCK$-ideal of $X$ for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$. Let $x, y \in X$ and $\omega \in \Omega$ be such that $x \in \xi(\omega)$. Since $X$ is a $d^*$-algebra, we have $(x \rightarrow y) \leadsto x = 0 \in \xi(\omega)$ and $(x \leadsto y) \rightarrow x = 0 \in \xi(\omega)$ for all $x, y \in X$. Using (b2), we get $x \rightarrow y \in \xi(\omega)$ and $x \leadsto y \in \xi(\omega)$ for all $y \in X$. Hence $\xi(\omega)$ is a pseudo $d$-ideal of $X$, and thus $\tilde{H}$ is a falling pseudo $d$-ideal of $X$. 

Corollary 3.9. In a pseudo $d^*$-algebra, every falling pseudo $BCK$-ideal is a falling pseudo $d$-subalgebra.

The following example shows that there exists a falling pseudo $d$-subalgebra which is not a falling pseudo $BCK$-ideal in pseudo $d^*$-algebras.

Example 3.10. Consider a pseudo $d^*$-algebra $X = \{0, a, b, c\}$ as in Example 3.7. For a probability space $(\Omega, A, P) = ([0, 1], A, m)$, define a random set

$$\xi : \Omega \rightarrow P(X), \quad \omega \mapsto \begin{cases} 
\{0, a, b\} & \text{if } t \in [0, 0.7), \\
X & \text{if } t \in [0.7, 0.8), \\
\emptyset & \text{if } t \in [0.8, 1].
\end{cases}$$

Then the falling shadow $\tilde{H}$ of $\xi$ is a falling pseudo $d$-subalgebra of $X$. If we take $t \in [0, 0.7)$, then $\xi(t) = \{0, a, b\}$ is not a pseudo $BCK$-ideal of $X$ since $c \rightarrow b = a, c \leadsto b = 0 \in \{0, a, b\}$ and $c \notin \{0, a, b\}$. Hence $\tilde{H}$ is not a falling pseudo $BCK$-ideal of $X$. 

Let $(\Omega, A, P)$ be a probability space and $\tilde{H}$ a falling shadow of a random set $\xi : \Omega \to \mathcal{P}(X)$. For any $x \in X$, let
\[ \Omega(x; \xi) := \{ \omega \in \Omega \mid x \in \xi(\omega) \} \]
Then $\Omega(x; \xi) \in A$.

**Lemma 3.11.** If $\tilde{H}$ is a falling pseudo $d$-subalgebra of $X$ then
\[ (\forall x \in X) \ (\Omega(x; \xi) \subseteq \Omega(0; \xi)) \]  
(3.1)

**Proof.** If $\Omega(x; \xi) = \emptyset$, then it is clear. Assume that $\Omega(x; \xi) \neq \emptyset$ and let $\omega \in \Omega$ be such that $\omega \in \Omega(x; \xi)$. Then $x \in \xi(\omega)$, and so $0 = x \to x = x \sim x \in \xi(\omega)$ since $\xi(\omega)$ is a pseudo $d$-subalgebra of $X$. Hence $\omega \in \Omega(0; \xi)$, and thus $\Omega(x; \xi) \subseteq \Omega(0; \xi)$ for all $x \in X$. \[\square\]

**Corollary 3.12.** If $\tilde{H}$ is a falling pseudo $d$-ideal of $X$, then (3.1) is valid.

**Theorem 3.13.** Let $\tilde{H}$ be a falling shadow of a random set $\xi$ on $X$. Then $\tilde{H}$ is a falling pseudo $d$-ideal of $X$ if and only if the following conditions are valid:

(a) $\ (\forall x, y \in X) \ (\Omega(x \to y; \xi) \cap \Omega(y; \xi) \cap \Omega(x \sim y; \xi) \subseteq \Omega(x; \xi))$,

(b) $\ (\forall x, y \in X) \ (\Omega(x; \xi) \subseteq \Omega(x \to y; \xi) \cap \Omega(x \sim y; \xi))$.

**Proof.** Suppose that $\tilde{H}$ is a falling pseudo $d$-ideal of $X$. Let $x, y \in X$ and $\omega \in \Omega$. If $\omega \in \Omega(x \to y; \xi) \cap \Omega(y; \xi) \cap \Omega(x \sim y; \xi)$, then $y \in \xi(\omega)$, $x \to y \in \xi(\omega)$ and $x \sim y \in \xi(\omega)$. Since $\xi(\omega)$ is a pseudo $d$-ideal of $X$, it follows from (b2) that $x \in \xi(\omega)$ so that $\omega \in \Omega(x; \xi)$. Hence
\[ \Omega(x \to y; \xi) \cap \Omega(y; \xi) \cap \Omega(x \sim y; \xi) \subseteq \Omega(x; \xi). \]

If $\omega \in \Omega(x; \xi)$, then $x \in \xi(\omega)$ and so $x \to y \in \xi(\omega)$ and $x \sim y \in \xi(\omega)$ by (b3). Hence $\omega \in \Omega(x \to y; \xi) \cap \Omega(x \sim y; \xi)$, and thus $\Omega(x; \xi) \subseteq \Omega(x \to y; \xi) \cap \Omega(x \sim y; \xi)$.

Conversely, suppose that two conditions (a) and (b) are valid. Let $x, y \in X$ and $\omega \in \Omega$ be such that $x \to y \in \xi(\omega)$, $x \sim y \in \xi(\omega)$ and $y \in \xi(\omega)$. Then $\omega \in \Omega(x \sim y; \xi)$, $\omega \in \Omega(x \sim y; \xi)$ and $\omega \in \Omega(y; \xi)$. It follows from (a) that
\[ \omega \in \Omega(x \to y; \xi) \cap \Omega(y; \xi) \cap \Omega(x \sim y; \xi) \subseteq \Omega(x; \xi) \]
so that $x \in \xi(\omega)$. Now assume that $x \in \xi(\omega)$ for every $x \in X$ and $\omega \in \Omega$. Then
\[ \omega \in \Omega(x; \xi) \subseteq \Omega(x \to y; \xi) \cap \Omega(x \sim y; \xi) \]
for all $y \in X$ by (b). Hence $x \to y \in \xi(\omega)$ and $x \sim y \in \xi(\omega)$ for all $y \in X$. Therefore $\tilde{H}$ is a falling pseudo $d$-ideal of $X$. \[\square\]
Corollary 3.14. For a falling shadow \( \tilde{H} \) of a random set \( \xi \) on \( X \), if the conditions (a) and (b) in Theorem 3.13 hold, then \( \tilde{H} \) is a falling pseudo \( d \)-ideal of \( X \).

Proposition 3.15. For a falling shadow \( \tilde{H} \) of a random set \( \xi \) on \( X \), if \( \tilde{H} \) is a falling pseudo \( BCK \)-ideal or a falling pseudo \( d \)-ideal of \( X \), then

\[
(\forall x, y \in X) \ (x \rightarrow y = 0 = x \rightsquigarrow y \Rightarrow \Omega(y; \xi) \subseteq \Omega(x; \xi)).
\]  

(3.2)

Proof. Assume that \( \tilde{H} \) is a falling pseudo \( d \)-ideal of \( X \). Let \( x, y \in X \) and \( \omega \in \Omega \) be such that \( x \rightarrow y = 0 = x \rightsquigarrow y \) and \( \omega \in \Omega(y; \xi) \). Then \( y \in \xi(\omega) \) and \( \omega \in \Omega(0; \xi) \) by Corollary 3.12. Hence \( x \rightarrow y = 0 \in \xi(\omega) \) and \( x \rightsquigarrow y = 0 \in \xi(\omega) \). Since \( \xi(\omega) \) is a pseudo \( d \)-ideal of \( X \), it follows from (b2) that \( x \in \xi(\omega) \) so that \( \omega \in \Omega(x; \xi) \). Therefore \( \Omega(y; \xi) \subseteq \Omega(x; \xi) \) for all \( x, y \in X \) with \( x \rightarrow y = 0 = x \rightsquigarrow y \).

Now suppose that \( \tilde{H} \) is a falling pseudo \( BCK \)-ideal of \( X \). Let \( x, y \in X \) and \( \omega \in \Omega \) be such that \( x \rightarrow y = 0 = x \rightsquigarrow y \) and \( \omega \in \Omega(y; \xi) \). Then \( y \in \xi(\omega) \) and \( x \rightarrow y = x \rightsquigarrow y = 0 \in \xi(\omega) \) by (b1). It follows from (b2) that \( x \in \xi(\omega) \) so that \( \omega \in \Omega(x; \xi) \). Hence \( \Omega(y; \xi) \subseteq \Omega(x; \xi) \) for all \( x, y \in X \) with \( x \rightarrow y = 0 = x \rightsquigarrow y \).

\( \square \)

Proposition 3.16. For a falling shadow \( \tilde{H} \) of a random set \( \xi \) on \( X \), if \( \tilde{H} \) is a falling pseudo \( BCK \)-ideal or a falling pseudo \( d \)-ideal of \( X \), then

\[
(\forall x, y \in X) \ (\Omega(x \rightarrow y; \xi) \cap \Omega(y; \xi) \cap \Omega(x \rightsquigarrow y; \xi) \subseteq \Omega(x; \xi)),
\]  

(3.3)

Proof. Let \( x, y \in X \) and \( \omega \in \Omega \) be such that \( \omega \in \Omega(x \rightarrow y; \xi) \cap \Omega(y; \xi) \cap \Omega(x \rightsquigarrow y; \xi) \). Then \( x \rightarrow y \in \xi(\omega) \), \( y \in \xi(\omega) \) and \( x \rightsquigarrow y \in \xi(\omega) \). Since \( \xi(\omega) \) is a pseudo \( BCK \)-ideal or a pseudo \( d \)-ideal of \( X \), it follows from (b2) that \( x \in \xi(\omega) \) so that \( \omega \in \Omega(x; \xi) \). Hence

\[
\Omega(x \rightarrow y; \xi) \cap \Omega(y; \xi) \cap \Omega(x \rightsquigarrow y; \xi) \subseteq \Omega(x; \xi)
\]

for all \( x, y \in X \).

\( \square \)

We give conditions for a falling shadow to be a falling pseudo \( BCK \)-ideal.

Theorem 3.17. For a falling shadow \( \tilde{H} \) of a random set \( \xi \) on \( X \), assume that the following conditions are satisfied:

(a) \( \Omega = \Omega(0; \xi) \),

(b) \( (\forall x, y \in X) (\Omega(x \rightarrow y; \xi) \cap \Omega(y; \xi) \cap \Omega(x \rightsquigarrow y; \xi) \subseteq \Omega(x; \xi)) \).

Then \( \tilde{H} \) is a falling pseudo \( BCK \)-ideal of \( X \).
Proof. Using (a), we have $0 \in \xi(\omega)$ for all $\omega \in \Omega$. Let $x, y \in X$ and $\omega \in \Omega$ be such that $x \rightarrow y \in \xi(\omega)$, $y \in \xi(\omega)$ and $x \sim y \in \xi(\omega)$. Then

$$\omega \in \Omega(x \rightarrow y; \xi) \cap \Omega(y; \xi) \cap \Omega(x \sim y; \xi) \subseteq \Omega(x; \xi)$$

by (b), and so $x \in \xi(\omega)$. Therefore $\xi(\omega)$ is a pseudo $BCK$-ideal of $X$ for all $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$. Hence $\tilde{H}$ is a falling pseudo $BCK$-ideal of $X$.

Corollary 3.18. For a falling shadow $\tilde{H}$ of a random set $\xi$ on $X$, assume that two conditions (a) and (b) in Theorem 3.17 are satisfied. If $X$ is a pseudo $d^*$-algebra, then $\tilde{H}$ is a falling pseudo $d$-ideal of $X$.

Theorem 3.19. If $\tilde{H}$ is a falling pseudo $d$-subalgebra of $X$, then

$$(\forall x, y \in X) \left( \min\{\tilde{H}(x \rightarrow y), \tilde{H}(x \sim y)\} \geq T_m(\tilde{H}(x), \tilde{H}(y)) \right) \tag{3.4}$$

where $T_m(s, t) = \max\{s + t - 1, 0\}$ for any $s, t \in [0, 1]$.

Proof. By Definition 3.1, $\xi(\omega)$ is a pseudo $d$-subalgebra of $X$ for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$. Hence

$$\{\omega \in \Omega \mid x \in \xi(\omega)\} \cap \{\omega \in \Omega \mid y \in \xi(\omega)\} \subseteq \{\omega \in \Omega \mid x \rightarrow y \in \xi(\omega)\}$$

and

$$\{\omega \in \Omega \mid x \in \xi(\omega)\} \cap \{\omega \in \Omega \mid y \in \xi(\omega)\} \subseteq \{\omega \in \Omega \mid x \sim y \in \xi(\omega)\}$$

which imply that

$$\tilde{H}(x \rightarrow y) = P(\omega \mid x \rightarrow y \in \xi(\omega))$$

$$\geq P(\{\omega \mid x \in \xi(\omega)\} \cap \{\omega \mid y \in \xi(\omega)\})$$

$$\geq P(\omega \mid x \in \xi(\omega)) + P(\omega \mid y \in \xi(\omega))$$

$$-P(\omega \mid x \in \xi(\omega) \text{ or } y \in \xi(\omega))$$

$$\geq \tilde{H}(x) + \tilde{H}(y) - 1$$

and

$$\tilde{H}(x \sim y) = P(\omega \mid x \sim y \in \xi(\omega))$$

$$\geq P(\{\omega \mid x \in \xi(\omega)\} \cap \{\omega \mid y \in \xi(\omega)\})$$

$$\geq P(\omega \mid x \in \xi(\omega)) + P(\omega \mid y \in \xi(\omega))$$

$$-P(\omega \mid x \in \xi(\omega) \text{ or } y \in \xi(\omega))$$

$$\geq \tilde{H}(x) + \tilde{H}(y) - 1.$$ 

Hence

$$\tilde{H}(x \rightarrow y) \geq \max\{\tilde{H}(x) + \tilde{H}(y) - 1, 0\} = T_m(\tilde{H}(x), \tilde{H}(y))$$

and

$$\tilde{H}(x \sim y) \geq \max\{\tilde{H}(x) + \tilde{H}(y) - 1, 0\} = T_m(\tilde{H}(x), \tilde{H}(y)).$$

Therefore $\min\{\tilde{H}(x \rightarrow y), \tilde{H}(x \sim y)\} \geq T_m(\tilde{H}(x), \tilde{H}(y))$ for all $x, y \in X$. \qed
Falling pseudo $d$-ideals and $d$-subalgebras in pseudo $d$-algebras

References


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