Zeros of the Second Kind  
(h, q)-Bernoulli Polynomials

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Abstract
In this paper, we investigate the zeros of the second kind (h, q)-Bernoulli polynomials \( B_{n,q}^{(h)}(x) \).

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1 Introduction
It is the aim of this paper to observe an interesting phenomenon of ‘scattering’ of the zeros of the second kind (h, q)-Bernoulli polynomials \( B_{n,q}^{(h)}(x) \) in complex plane. The outline of this paper is as follows. We introduce the second kind (h, q)-Bernoulli numbers \( B_{n,q}^{(h)} \) and polynomials \( B_{n,q}^{(h)}(x) \). We give some properties of these numbers \( B_{n,q}^{(h)} \) and polynomials \( B_{n,q}^{(h)}(x) \). In Section 2, we describe the beautiful zeros of the second kind (h, q)-Bernoulli polynomials \( B_{n,q}^{(h)}(x) \) using a numerical investigation. Finally, we investigate the roots of the second kind (h, q)-Bernoulli polynomials \( B_{n,q}^{(h)}(x) \). Throughout this paper, we always make use of the following notations: \( \mathbb{N} = \{1, 2, 3, \cdots \} \) denotes the set of natural numbers, \( \mathbb{Z} \) denotes the set of integers, \( \mathbb{R} \) denotes the set of real numbers, and \( \mathbb{C} \) denotes the set of complex numbers.

Let \( q \) be a complex number with \( |q| < 1 \) and \( h \in \mathbb{Z} \). By the meaning of (1.1) and (1.2), let us define the second kind (h, q)-Bernoulli numbers \( B_{n,q}^{(h)} \) and polynomials \( B_{n,q}^{(h)}(x) \) as follows:

\[
F_q^{(h)}(t) = \frac{2te^t}{q^h e^{2t} - 1} = \sum_{n=0}^{\infty} B_{n,q}^{(h)} \frac{t^n}{n!},
\]  
(1.1)
The following elementary properties of the second kind \((h, q)\)-Bernoulli polynomials \(B_{n,q}^{(h)}(x)\) are readily derived from (1.1) and (1.2). We, therefore, choose to omit the details involved. More studies and results in this subject we may see references [1]-[11].

**Theorem 1.1** For any positive integer \(n\), we have

\[
q^h B_{n,q}^{(h)}(x + 2) - B_{n,q}^{(h)}(x) = 2n(x + 1)^{n-1}. \tag{1.3}
\]

**Theorem 1.2** For \(n \in \mathbb{N}\), we have

\[
B_{n,q}^{(h)}(x) = (-1)^n q^{-h} B_{n,q-1}^{(h)}(-x). \tag{1.4}
\]

By (1.3) and (1.4), we have

\[
B_{n,q}^{(h)}(-1) = (-1)^n B_{n,q-1}^{(h)}(-1).
\]

2 **Zeros of the second kind \((h, q)\)-Bernoulli polynomials \(B_{n,q}^{(h)}(x)\)**

In this section, we assume that \(q \in \mathbb{C}\) with \(0 < q < 1\). We investigate the beautiful zeros of the second kind \((h, q)\)-Bernoulli polynomials \(B_{n,q}^{(h)}(x)\) by using a computer. We plot the zeros of the second kind \((h, q)\)-Bernoulli polynomials \(B_{n,q}^{(h)}(x)\) for \(n = 10, 20, 25, 30, q = 1/10, h = 4\) and \(x \in \mathbb{C}\) (Figure 1). In Figure 1(top-left), we choose \(n = 10, q = 1/2\) and \(h = 4\). In Figure 1(top-right), we choose \(n = 20, q = 1/2\) and \(h = 4\). In Figure 1(bottom-left), we choose \(n = 25, q = 1/2\) and \(h = 4\). In Figure 1(bottom-right), we choose \(n = 30, q = 1/2\) and \(h = 4\).

We plot the zeros of the second kind \((h, q)\)-Bernoulli polynomials \(B_{n,q}^{(h)}(x)\) for \(n = 30, q = 1/10, h = 10, 15, 20, 25\) and \(x \in \mathbb{C}\) (Figure 2). In Figure 2(top-left), we choose \(n = 30, q = 1/2\) and \(h = 10\). In Figure 2(top-right), we choose \(n = 30, q = 1/2\) and \(h = 15\). In Figure 2(bottom-left), we choose \(n = 30, q = 1/2\) and \(h = 20\). In Figure 2(bottom-right), we choose \(n = 30, q = 1/2\) and \(h = 25\).

Stacks of zeros of \(B_{n,q}^{(h)}(x)\) for \(1 \leq n \leq 30\) from a 3-D structure are presented (Figure 3). In Figure 3, we choose \(q = 1/2\) and \(h = 4\). Our numerical results for approximate solutions of real zeros of \(B_{n,q}^{(h)}(x), q = 1/2\) are displayed(Tables 1, 2).
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We observe a remarkably regular structure of the complex roots of the second kind \((h, q)\)-Bernoulli polynomials \(B_{n,q}^{(h)}(x)\). We hope to verify a remarkably regular structure of the complex roots of the second kind \((h, q)\)-Bernoulli polynomials \(B_{n,q}^{(h)}(x)\)(Table 1). Next, we calculated an approximate solution satisfying \(B_{n,q}^{(h)}(x)\) for \(q = 1/2\) and \(x \in \mathbb{R}\). The results are given in Table 2.
Figure 2: Zeros of $B^{(h)}_{n,q}(x)$ for $h = 10, 15, 20, 25$

<table>
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<td>20</td>
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</table>
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Figure 3: Stacks of zeros of \(B^{(h)}_{n,q}(x)\), \(1 \leq n \leq 30\)

Figure 4 shows the distribution of real zeros of \(B^{(h)}_{n,q}(x)\) for \(1 \leq n \leq 30\) and \(h = 4\). We want to find a formula that best fits a given set of data points. The least squares method is used to fit a polynomials or a set of functions to a given set of data points. Using the least squares method, we can find \(a\) and \(b\) such that \(x = a + bn\) is the least squares fit to the data given in Table 2. The graph of the data points is shown in Figure 4. We obtain \(x = -0.8476559934976322 - 0.20891403938843115n\). That is, the result is the best linear combination of the function 2 and \(n\). The real zero \(x \sim -\infty\) asymptotically as \(n \to \infty\).

The plot above shows \(B^{(h)}_{n,q}(x)\) for real \(-9/10 \leq q \leq 9/10\) and \(-3 \leq x \leq 3\), with the zero contour indicated in black(Figure 5). In Figure 5(top-left), we choose \(n = 2\). In Figure 5(top-right), we choose \(n = 4\). In Figure 5(bottom-
Finally, we shall consider the more general problems. Prove that $B_{n,q}(x), x \in \mathbb{C}$, has $Re(x) = 0$ reflection symmetry in addition to the usual $Im(x) = 0$ reflection symmetry analytic complex functions (see Figures 1, 2, 3). The obvious corollary is that the zeros of $B_{n,q}(x)$ will also inherit these symmetries.

If $B_{n,q}(x_0) = 0$, then $B_{n,q}(x_0^*) = 0$.

* denotes complex conjugation (see Figures 1, 2). Prove that $B_{n,q}(x) = 0$ has $n - 1$ distinct solutions. Find the numbers of complex zeros $C_{B_{n,q}(x)}$ of $B_{n,q}(x), Im(x) \neq 0$. Since $n - 1$ is the degree of the polynomial $B_{n,q}(x)$, the number of real zeros $R_{B_{n,q}(x)}$ lying on the real plane $Im(x) = 0$ is then $R_{B_{n,q}(x)} = n - 1 - C_{B_{n,q}(x)}$, where $C_{B_{n,q}(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{B_{n,q}(x)}$ and $C_{B_{n,q}(x)}$. For related topics the interested reader is referred to [8, 9, 10, 11].

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References


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