Some Properties Related to
the Generalized $q$-Genocchi Numbers
and Polynomials with Weak Weight $\alpha$

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Abstract

Recently many mathematicians are working on Genocchi numbers and Genocchi polynomials. We define a new generalized $q$-Genocchi numbers $G^{(\alpha)}_{n,\lambda,q}$ and polynomials $G^{(\alpha)}_{n,\lambda,q}(x)$ with weak weight $\alpha$ and give some interesting relations of their numbers and polynomials with weak weight $\alpha$. Also, we construct generalized $q$-Genocchi zeta function and generalized Hurwitz $q$-Genocchi zeta function and find relations between generalized $q$-Genocchi numbers and polynomials with weak weight $\alpha$ and their zeta functions.

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1 Introduction

Many mathematicians defined the generalized $q$-Genocchi numbers and polynomials by using $p$-adic invariant integrals on $\mathbb{Z}_p$(see [1-9]). Also they introduced generalized $q$-Genocchi zeta function which interpolate $q$-Genocchi polynomials, in [1,3,4,6,9]. In the paper, our aim is to construct the new generalized $q$-Genocchi numbers $G^{(\alpha)}_{n,\lambda,q}$ and polynomials $G^{(\alpha)}_{n,\lambda,q}(x)$ with weak weight $\alpha$ by
using \( q \)-volkenborn integration. Next we construct new generalized \( q \)-Genocchi zeta function and new generalized Hurwitz \( q \)-Genocchi function which interpolate the generalized \( q \)-Genocchi numbers and polynomials with weak weight \( \alpha \) at negative integer. Throughout this paper we use the following notations.

Let \( v_p \) of numbers, the complex number field, and the completion of the algebraic closure normally assumes this paper, we use the following notation:

Hence \( \lim_{q \to 1} [x]_q = x \) for all \( x \in \mathbb{Z}_p \). For \( g \in UD(\mathbb{Z}_p) \), Kim defined the \( q \)-deformed fermionic \( p \)-adic integral on \( \mathbb{Z}_p \)

\[
I_{−q}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{−q}(x) = \lim_{N \to \infty} \frac{1}{[p^{N}]_{−q}} \sum_{0 < a < dp} g(x)(−q)^{x}. \tag{1.2}
\]

Let \( g_1(x) \) be the translation with \( g_1(x) = g(x + 1) \). Then we have the following integral equation:

\[
q^n I_{−q}(g_n) + (−1)^{n−1} I_{−q}(g) = [2]_q \sum_{l=0}^{n−1} (−1)^{n−1−l} q^l. \tag{1.3}
\]

For \( g \in UD(\mathbb{Z}_p) \),

\[
\int_{\mathbb{Z}_p} g(x) d\mu_{−q}(x) = \int_{\mathbb{X}} g(x) d\mu_{−q}(x). \tag{1.4}
\]

We introduced generalized Genocchi number and polynomials. Let \( \chi \) be a primitive Dirichlet character of conductor \( f \in \mathbb{N} \). We assume that \( f \) is odd. Then the generalized Genocchi numbers associated with \( \chi \) are defined by

\[
F_{\chi}(t) = \frac{2t \sum_{i=0}^{f−1} \chi(i)(−1)^i e^{iit}}{e^{ft} + 1} = \sum_{n=0}^{\infty} G_{n,\chi} \frac{t^n}{n!}. \tag{1.5}
\]

The generalized Genocchi polynomials associated with \( \chi \) are also defined by

\[
F_{\chi}(t, x) = \frac{2t \sum_{i=0}^{f−1} \chi(i)(−1)^i e^{iit}}{e^{ft} + 1} e^{tx} = \sum_{n=0}^{\infty} G_{n,\chi}(x) \frac{t^n}{n!}. \tag{1.6}
\]
In the special case $x = 0$, $G_{n,\chi} = G_{n,\chi}(0)$ are called the $n$-th generalized Genocchi numbers attached to $\chi$. These numbers and polynomials are interpolated by the $q$-Genocchi zeta function and Hurwitz $q$-Genocchi zeta function, respectively.

2 Generalized $q$-Genocchi numbers and polynomials with weak weight $\alpha$

Our primary goal of this section is to define the generalized $q$-Genocchi numbers and polynomials with weak weight $\alpha$. We also find generating functions of the generalized $q$-Genocchi numbers and polynomials with weak weight $\alpha$. These polynomials will be used to prove the analytic continuation of the generalized Hurwitz $q$-Genocchi zeta function. First, we introduce the generalized $q$-Genocchi numbers with weak weight $\alpha$.

**Definition 2.1** For $q \in \mathbb{C}_p$ with $|1-q|_p < 1$, $\alpha \in \mathbb{Z}$,

$$G^{(\alpha)}_{n,\chi,q} = \int_X \chi(x)n[x]_{q}^{n-1}d\mu_{-q^\alpha}(x). \quad (2.1)$$

By using $p$-adic $q$-integral, we have

$$\int_X \chi(x)n[x]_{q}^{n-1}d\mu_{-q^\alpha}(x)
= [2]_q^n \sum_{i=0}^{f-1} (-1)^i q^{i\alpha} \chi(i) \left( \frac{1}{1-q} \right)^{n-1} \sum_{l=0}^{n-1} \left( \frac{n-1}{l} \right) (-1)^l q^l \frac{1}{1+q^{(\alpha+l)f}}. \quad (2.2)$$

By (2.1), we obtain

$$G^{(\alpha)}_{n,\chi,q} = [2]_q^n \sum_{i=0}^{f-1} (-1)^i q^{i\alpha} \chi(i) \left( \frac{1}{1-q} \right)^{n-1} \sum_{l=0}^{n-1} \left( \frac{n-1}{l} \right) (-1)^l q^l \frac{1}{1+q^{(\alpha+l)f}}. \quad (2.3)$$

In order to find the generating function of $G^{(\alpha)}_{n,\chi,q}$, we set

$$F^{(\alpha)}_{\chi,q}(t) = \sum_{n=0}^{\infty} G^{(\alpha)}_{n,\chi,q} \frac{t^n}{n!}. \quad (2.4)$$

By using (2.3), we have

$$G^{(\alpha)}_{n,\chi,q} = [2]_q^n \sum_{i=0}^{f-1} (-1)^i q^{i\alpha} \chi(i) \left( \frac{1}{1-q} \right)^{n-1} \sum_{l=0}^{n-1} \left( \frac{n-1}{l} \right) (-1)^l q^l \frac{1}{1+q^{(\alpha+l)f}}
= [2]_q^n \sum_{l=0}^{\infty} (-1)^l q^{\alpha l} \chi(l)[l]_q^{n-1}. \quad (2.5)$$
By using (2.4) and (2.5), we obtain that
\[
F_{\chi,q}^{(\alpha)}(t) = \sum_{n=0}^{\infty} \left[2\right]_{q^\alpha} t^n \sum_{l=0}^{\infty} (-1)^l q^\alpha (l)[l]_q^{n-1} \frac{t^n}{n!} = \left[2\right]_{q^\alpha} t \sum_{n=0}^{\infty} (-1)^l q^\alpha (l)e^{[l]_q t}.
\] (2.6)

Then generalized \(q\)-Genocchi numbers \(G_{n,\chi,q}^{(\alpha)}\) with weak weight \(\alpha\) are defined by means of the generating function
\[
F_{\chi,q}^{(\alpha)}(t) = \left[2\right]_{q^\alpha} t \sum_{n=0}^{\infty} (-1)^n q^\alpha n! = \sum_{n=0}^{\infty} G_{n,\chi,q}^{(\alpha)} \frac{t^n}{n!}.
\] (2.7)

**Remark 2.2** In (2.7), we see that
\[
\lim_{q \to 1} F_{\chi,q}^{(\alpha)}(t) = 2t \sum_{n=0}^{\infty} (-1)^n \chi(n)e^{nt} = \frac{2t \sum_{i=0}^{f-1} \chi(i)(-1)^i e^{it}}{e^{it} + 1} = F_{\chi}(t).
\] (2.8)

By using (2.1), we obtain
\[
\sum_{n=0}^{\infty} G_{n,\chi,q}^{(\alpha)} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \int_X \chi(x)n[x]_q^{n-1} d\mu_{-q^\alpha}(x) t^n
\] (2.9)

By (2.7) and (2.9), we have
\[
\int_X t\chi(x)e^{[x]_q t} d\mu_{-q^\alpha}(x) = \left[2\right]_{q^\alpha} t \sum_{n=0}^{\infty} (-1)^n q^\alpha n! \chi(n)e^{[n]_q t}.
\] (2.10)

Next, we introduce the generalized \(q\)-Genocchi polynomials with weak weight \(\alpha\).

**Definition 2.3** For \(q \in \mathbb{C}_p\) with \(|1 - q|_p < 1\), \(\alpha \in \mathbb{Z}\),
\[
G_{n,\chi,q}^{(\alpha)}(x) = \int_X \chi(y)n[x + y]_q^{n-1} d\mu_{-q^\alpha}(y).
\] (2.11)

By using \(p\)-adic \(q\)-integral, we have
\[
\int_X \chi(y)n[x + y]_q^{n-1} d\mu_{-q^\alpha}(y)
\] (2.12)
By using (2.11) and (2.12), we get

\[
G_{n,\chi,q}^{(\alpha)}(x) = [2]q^{\alpha}n \sum_{i=0}^{f-1} (-1)^{i}q^{\alpha}(i) \left( \frac{1}{1-q} \right)^{n} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^{l}q^{\alpha} \chi(i) \left( \frac{1}{1+q^{\alpha+l}} \right).
\]  

(2.13)

In order to find the generating function of \( G_{n,\chi,q}^{(\alpha)}(x) \), we set

\[
F_{\chi,q}^{(\alpha)}(t, x) = \sum_{n=0}^{\infty} G_{n,\chi,q}^{(\alpha)}(x) \frac{t^{n}}{n!}.
\]  

(2.14)

By using (2.13), we obtain

\[
G_{n,\chi,q}^{(\alpha)}(x) = [2]q^{\alpha} \sum_{l=0}^{\infty} (-1)^{l}q^{\alpha l} \chi(l)[x + l]_{q}^{n-1}.
\]  

(2.15)

By using (2.14), we get

\[
F_{\chi,q}^{(\alpha)}(t, x) = \sum_{n=0}^{\infty} \left( [2]q^{\alpha} \sum_{l=0}^{\infty} (-1)^{l}q^{\alpha l} \chi(l)[x + l]_{q}^{n-1} \right) \frac{t^{n}}{n!}
\]

\[
= [2]q^{\alpha} t \sum_{l=0}^{\infty} (-1)^{l}q^{\alpha l} \chi(l)e^{[x+l]_{q}t}.
\]

(2.16)

Hence, we are defined the generating function of \( G_{n,\chi,q}^{(\alpha)}(x) \)

\[
F_{\chi,q}^{(\alpha)}(t, x) = [2]q^{\alpha} t \sum_{n=0}^{\infty} (-1)^{n}q^{\alpha n} \chi(n)e^{[x+n]_{q}t} = \sum_{n=0}^{\infty} G_{n,\chi,q}^{(\alpha)}(x) \frac{t^{n}}{n!}.
\]  

(2.17)

**Remark 2.4** In (2.17), we observe that

\[
\lim_{q \to 1} F_{\chi,q}^{(\alpha)}(t, x) = 2t \sum_{n=0}^{\infty} (-1)^{n} \chi(n)e^{(x+n)t}
\]

\[
= \frac{2t \sum_{i=0}^{f-1} \chi(i)(-1)^{i}e^{it}}{e^{ft+1}e^{xt}} = F_{\chi}(t, x).
\]  

(2.18)
3 Some relations related to $G_{n,\chi,q}^{(\alpha)}$ and $G_{n,\chi,q}^{(\alpha)}(x)$

In this section, we investigate some relations related to $G_{n,\chi,q}^{(\alpha)}$ and $G_{n,\chi,q}^{(\alpha)}(x)$.

Since $[x + y]_q = [x]_q + q^x[y]_q$, we have

$$G_{n+1,\chi,q}^{(\alpha)}(x) = (n + 1) \int_X \chi(y)[x + y]_q^n d\mu_{q^\alpha}(y) = q^{-x}([x]_q + q^x G_{\chi,q}^{(\alpha)})^{n+1}. \quad (3.1)$$

Also, we get

$$\sum_{n=0}^{\infty} G_{n,\chi,q}^{(\alpha)}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \int_X \chi(y)n \sum_{l=0}^{n-1} \binom{n-1}{l} [x]_q^{n-1-l} q^x[y]_q^l d\mu_{q^\alpha}(y) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( n \sum_{l=0}^{n-1} \binom{n-1}{l} [x]_q^{n-1-l} q^x \frac{G_{n+1,\chi,q}^{(\alpha)}}{l+1} \right) \frac{t^n}{n!}. \quad (3.2)$$

By comparing coefficient $\frac{t^n}{n!}$ in (3.2), we have the following theorem.

**Theorem 3.1** Let $n \in \mathbb{N}$. Then we have

$$G_{n,\chi,q}^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k} [x]_q^{n-k} q^x(k-1) G_{k,\chi,q}^{(\alpha)}. \quad (3.3)$$

By using (2.1) and (2.11), we have the following theorem.

**Theorem 3.2** Let $n \in \mathbb{N}$. We have

$$G_{n,\chi,q}^{(\alpha)} = \frac{[f]_q^{n-1}}{[f]_q - q^\alpha} \sum_{i=0}^{f-1} \chi(i)(-1)^i q^{i\alpha} G_{n,q^i,f}^{(\alpha)}(\frac{i}{f}),$$

$$G_{n,\chi,q}^{(\alpha)}(x) = \frac{[f]_q^{n-1}}{[f]_q - q^\alpha} \sum_{i=0}^{f-1} \chi(i)(-1)^i q^{i\alpha} G_{n,q^i,f}^{(\alpha)}(\frac{i+x}{f}).$$

By using (1.7), we easily see that

$$q^{m[nf]} G_{m,\chi,q}^{(\alpha)}(nf) + (-1)^{nf-1} G_{m,\chi,q}^{(\alpha)} = [2][q^\alpha m \sum_{l=0}^{nf-1} (-1)^{nf-1-l} q^l \chi(l)]_q^{m-1}. \quad (3.4)$$

Hence, we have the following theorem.
Theorem 3.3 Let \( m \in \mathbb{Z}^+ \). If \( n \equiv 0 \pmod{2} \), then
\[
q^{\alpha nf}G_{m,\chi,q}^{(a)}(nf) - G_{m,\chi,q}^{(a)} = [2]_{q^\alpha} m \sum_{l=0}^{nf-1} (-1)^{l+1} q^{al} \chi(l) [l]_q^{m-1},
\]
If \( n \equiv 1 \pmod{2} \), then
\[
q^{\alpha nf}G_{m,\chi,q}^{(a)}(nf) + G_{m,\chi,q}^{(a)} = [2]_{q^\alpha} m \sum_{l=0}^{nf-1} (-1)^{l} q^{al} \chi(l) [l]_q^{m-1}.
\]

4 The \( q \)-Genocchi zeta function

In the section, we assume that \( q \in \mathbb{C} \) with \(|q| < 1\). By using generalized \( q \)-Genocchi numbers and polynomials with weak weight \( \alpha \), generalized \( q \)-Genocchi zeta function and generalized Hurwitz \( q \)-Genocchi zeta function are defined. These functions interpolate the generalized \( q \)-Genocchi numbers with weak weight \( \alpha \) and the generalized \( q \)-Genocchi polynomials with weak weight \( \alpha \), respectively.

From (2.7), we note that
\[
\frac{d^{k+1}}{dt^{k+1}} E_{\chi,q}^{(\alpha)}(t) \bigg|_{t=0} = [2]_{q^\alpha} (k+1) \sum_{n=0}^{\infty} (-1)^n q^{an} \chi(n) [n]_q^k, (k \in \mathbb{N}). \tag{4.1}
\]

Definition 4.1 For \( s \in \mathbb{C} \), we define
\[
\zeta_{\chi,q}^{(\alpha)}(s) = [2]_{q^\alpha} \sum_{n=1}^{\infty} \frac{(-1)^n q^{an} \chi(n)}{[n]_q^s}. \tag{4.2}
\]
Note that \( \zeta_{\chi,q}^{(\alpha)}(s) \) is a meromorphic function on \( \mathbb{C} \).

Remark 4.2 Let \( s \in \mathbb{C} \). Then we have
\[
\lim_{q \to 1} \zeta_{\chi,q}^{(\alpha)}(s) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n \chi(n)}{n^s}. \tag{4.3}
\]
Relation between \( \zeta_{\chi,q}^{(\alpha)}(s) \) and \( G_{k,\chi,q}^{(a)} \) is given by the following theorem.

Theorem 4.3 For \( k \in \mathbb{N} \), we have
\[
\zeta_{\chi,q}^{(\alpha)}(-k) = \frac{1}{k+1} G_{k+1,\chi,q}^{(a)}. \tag{4.4}
\]
Observe that $\zeta^{(\alpha)}_{\chi,q}(s)$ interpolates $G^{(\alpha)}_{k,\chi,q}$ at non-negative integers.

By using (2.17), we note that

$$\frac{d^{k+1}}{dt^{k+1}} F^{(\alpha)}_{\chi,q}(t, x) \bigg|_{t=0} = [2]_{q^n}(k + 1) \sum_{l=0}^{\infty} (-1)^l q^{\alpha l} \chi(l)[x + l]_q^s, (k \in \mathbb{N}).$$  \hspace{1cm} (4.5)

By (4.5), we are now ready to define the generalized Hurwitz $q$- Genocchi zeta functions.

**Definition 4.4** Let $s \in \mathbb{C}$. Then we have

$$\zeta^{(\alpha)}_{\chi,q}(s, x) = [2]_{q^n} \sum_{l=0}^{\infty} (-1)^l q^{\alpha l} \chi(l) \frac{x + l}{{[x + l]_q}^s}. \hspace{1cm} (4.6)$$

Note that $\zeta^{(\alpha)}_{\chi,q}(s, x)$ is a meromorphic function on $\mathbb{C}$. Relation between $\zeta^{(\alpha)}_{\chi,q}(s, x)$ and $G^{(\alpha)}_{k,\chi,q}(x)$ is given by the following theorem.

**Theorem 4.5** For $k \in \mathbb{N}$, we get

$$\zeta^{(\alpha)}_{\chi,q}(-k, x) = \frac{1}{k + 1} G^{(\alpha)}_{k+1,\chi,q}(x). \hspace{1cm} (4.7)$$

Observe that $\zeta^{(\alpha)}_{\chi,q}(-k, x)$ function interpolates $G^{(\alpha)}_{k+1,\chi,q}(x)$ at non-negative integers.

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**References**


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