Numerical Solution of the (2+1)-Dimensional Boussinesq Equation with Initial Condition by Homotopy Perturbation Method

Ghanmi Imed

Faculte des Sciences de Tunis, campus universitaire, 1060
Tunis, Tunisia
imedghanmi07@yahoo.fr

Boukricha Abderrahmen

Faculte des Sciences de Tunis, campus universitaire, 1060
Tunis, Tunisia
aboukricha@fst.rnu.tn

Abstract

In this paper, an application of homotopy perturbation method is to solve the nonlinear (2+1)-dimensional Boussinesq Equation. The analytic of the nonlinear problem is calculated in the form of a series with easily computable components. The numerical solutions are compared with the known analytical solutions. The results prove that HPM is very effective and simple.

Keywords: homotopy perturbation method; (2+1)-dimensional Boussinesq Equation

1 Introduction

The partial differential equation (PDEs) play an important role in the physical sciences and in engineering fields. In recent years, the studies of the exact solutions for the nonlinear evolution equation have attracted the attention of many mathematicians and physicists [2-7]. Recently, Chen and AL.[9] studied the (2+1)-dimensional Boussinesq Equation by generalized transformation in homogeneous balance method [8] and obtained certain new solitary wave solution; periodic wave solution and the combined formal solitary wave solution and periodic wave solution of the equation. In [10-16], ELP-sayed and Kaya are
implemented the Adomian’s decomposition for obtaining explicit solutions of the nonlinear partial differential equations.

Homotopy perturbation method has been used by many mathematicians and engineers to solve various functional equations. This method was further developed and improved by He and applied to nonlinear oscillators with discontinuities [19], nonlinear wave equation [17], and boundary value problems [15]. It can be said that He’s homotopy perturbation method is a universal one and is able to solve various kinds of nonlinear functional equations. In this method the solution is considered as the summation of an infinite series which usually converges rapidly to exact solutions. This method continuously deforms the difficult equation under study into a simple equation, easy to solve.

For that we will study the (2+1)-dimensional Boussinesq Equation by using the homotopy perturbation method.

2 Basic Idea of homotopy perturbation method

The homotopy perturbation method is a combination of the classical perturbation technique and homotopy technique to illustrate HPM consider the following nonlinear differential equation

\[ A(u) - f(r) = 0 ; \quad r \in \Omega \]

subject to boundary condition

\[ B(u, \frac{\partial u}{\partial n}) = 0 ; r \in \Gamma \]

where \( A \) is a general differential operator, \( B \) a boundary operator, \( f(r) \) is a known analytic function, \( \Gamma \) is the boundary of the domain \( \Omega \) and \( \frac{\partial u}{\partial n} \) denotes differentiation along the normal drawn outward from \( \Omega \).

The operator \( A \) can be generally divided into two parts \( L \) and \( N \), where \( L \) is linear, whereas \( N \) is nonlinear. Equation (1) can be rewritten as follows

\[ L(u) + N(u) - f(r) = 0 \]

By using homotopy technique, we can construct a homotopy

\[ \nu : \Omega \times [0,1] \rightarrow \mathbb{R} \]

which satisfies

\[ H(\nu, p) = (1 - p)[L(\nu) - L(u_0)] + p[A(\nu) - f(r)] = 0 \]

which is equivalent to

\[ H(\nu, p) = L(\nu) - L(u_0) + pL(u_0) + p[N(\nu) - f(r)] = 0 \]
where \( p \in [0, 1] \) is an embedding parameter and \( u_0 \) is an arbitrary initial approximation of equation [1] which satisfies the boundary conditions. When we put \( p = 0 \) and \( p = 1 \) in equation (4), we obtain

\[
H(\nu, 0) = L(\nu) - L(u_0) = 0
\]

and

\[
H(\nu, 1) = A(\nu) - f(r) = 0
\]

and the changing process of \( p \) from 0 to 1; is just that of \( H(\nu, p) \) from \( L(\nu) - L(u_0) \) to \( A(\nu) - f(r) \). In topology this called deformation; \( L(\nu) - L(0) \) and \( A(\nu) - f(r) \) are called homotopic. Applying the perturbation technique [16], due to the fact that \( 0 \leq p \leq 1 \) can be considered as a small parameter, we can assume that the solution of (4) or (5) can be expressed as a series in \( p \), as follows:

\[
\nu = \nu_0 + p\nu_1 + p^2\nu_2 + p^3\nu_3 + .......
\]  

When \( p \to 1 \), (4) or (5) corresponds to (3) and becomes the approximate solution of (3):

\[
u = \lim_{p \to 1} \nu = \nu_0 + \nu_1 + \nu_2 + \nu_3 + .......
\]  

The series (9) is convergent for most cases; and the rate of convergence depends on \( A(\nu) \) [17].

The combination of a small parameter (perturbation parameter), with a homotopy is called the HPM as presented in the series (9). The convergence of the series (9) has been proved by He in his paper [18].

3 Solution of the Boussinesq equation by HPM

Consider the following nonlinear differential equation; with the indicated initial conditions:

\[
u_{tt} - u_{xx} - u_{yy} - (u^2)_{xx} - u_{xxxx} = \phi(x, y, t)
\]

\[
u(x, 0, t) = f(x, t)
\]

\[rac{\partial u}{\partial y}(x, 0, t) = g(x, t)
\]

for solving this equation by HPM, we rewrite eq(10) as follows:

\[
L_y(u) - N(u) = \phi(x, y, t)
\]

where the linear operator \( L_y = \frac{\partial^2}{\partial^2 y} \); with the inverse operator

\[
L_y^{-1} = \int_{0}^{y} \int_{0}^{z} (.)dxdy
\]
and
\[ N(u) = u_{tt} - u_{xx} - (u^2)_{xx} - u_{xxxx} \]  \quad (15)
represents the nonlinear term. The inverse operator, we get
\[ u(x, y, t) = f(x, t) + yg(x, t) + \int_0^y \int_0^z \phi(x, s, t) ds dz + \int_0^y \int_0^z N(x, s, t) ds dz \]  \quad (16)
By HPM \( L(u) = u(x, y, t) - h(x, y, t) = 0 \) where
\[ h(x, y, t) = f(x, t) + yg(x, t) + \int_0^y \int_0^z \phi(x, s, t) ds dz \]  \quad (17)
Hence, we may choose a convex homotopy such that
\[ H(\nu, p) = \nu(x, y, t) - h(x, y, t) - p \int_0^y \int_0^z u_{tt} - u_{xx} - (u^2)_{xx} - u_{xxxx} ds dz \]  \quad (18)
Substituting (8) into (18), and equating the terms with identical powers of \( p \), we have
\[ P_0^0 : \nu_0(x, y, t) = h(x, y, t) \]  \quad (19)
\[ P_1^1 : \nu_1 = \int_0^y \int_0^z \frac{\partial^2 \nu_0(x, s, t)}{\partial t^2} - \frac{\partial^2 \nu_0(x, s, t)}{\partial x^2} - \frac{\partial^2 \nu_0(x, s, t)^2}{\partial x^2} - \frac{\partial^4 \nu_0(x, s, t)}{\partial x^4} ds dz \]  \quad (20)
\[ P_2^2 : \nu_2 = \int_0^y \int_0^z \frac{\partial^2 \nu_1(x, s, t)}{\partial t^2} - \frac{\partial^2 \nu_1(x, s, t)}{\partial x^2} - 2 \frac{\partial^2 \nu_0(x, s, t) \nu_1(x, s, t)}{\partial x^2} - \frac{\partial^4 \nu_1(x, s, t)}{\partial x^4} ds dz \]  \quad (21)
\[ P_3^3 : \nu_3 = \int_0^y \int_0^z \frac{\partial^2 \nu_2(x, s, t)}{\partial t^2} - \frac{\partial^2 \nu_2(x, s, t)}{\partial x^2} - 2 \frac{\partial^2 \nu_0(x, s, t) \nu_2(x, s, t)}{\partial x^2} - \frac{\partial^2 \nu_1(x, s, t)^2}{\partial x^2} - \frac{\partial^4 \nu_2(x, s, t)}{\partial x^4} ds dz \]  \quad (22)
To give a clear overview of the methodology; as an illustrative example, we take (2+1)-dimensional Boussinesq equation described by
\[ u_{tt} - u_{xx} - u_{yy} - (u^2)_{xx} - u_{xxxx} = 0 \]  \quad (23)
\[ u(x, 0, t) = K - 6\alpha^2 \tanh^2(\alpha x - \gamma t) \]  \quad (24)
\[ \frac{\partial u}{\partial y}(x, 0, t) = -12\alpha^2 \beta \sech^2(\alpha x - \gamma t) \tanh(\alpha x - \gamma t) \]  \quad (25)
Consequently, solving the Boussinesq equation, the first few components of the homotopy perturbation method are derived as follows:

\[ \nu_0(x, y, t) = k - 6 \alpha^2 (\tanh (\alpha x + \gamma t))^2 - 12 y \alpha^2 \beta (\text{sech} (\alpha x + \gamma t))^2 \tanh (\alpha x + \gamma t) \] (26)

\[
\begin{align*}
u_1(x, y, t) &= \frac{3\alpha^2 y^2}{2}[3\alpha^2 - 3\gamma^2 - 132\alpha^4 + 2\alpha^2 \cosh(2\eta)] \\
&\quad - 2\gamma^2 \cosh(2\eta) + 104\alpha^4 \cosh(2\eta) - \alpha^2 \cosh(4\eta) \\
&\quad + \gamma^2 \cosh(4\eta) - 4\alpha^4 \cosh(4\eta)] \text{sech}(\eta)^6 \\
&\quad + \alpha^2 \gamma^2 (-9\alpha^2 + 9\gamma^2 + 492\alpha^4 - 8\alpha^2 \cosh(2\eta) + 8\gamma^2 \cosh(2\eta) \\
&\quad - 224\alpha^4 \cosh(2\eta) + \alpha^2 \cosh(4\eta) - \gamma^2 \cosh(4\eta) \\
&\quad + \alpha^4 \cosh(4\eta)] \text{sech}(\eta)^6 \tanh(\eta) \] (27)

\[
\begin{align*}
u_2(x, y, t) &= 3\alpha^4 y^2 [3K + 543\alpha^2 + 2K \cosh(2\eta) - 60\alpha^2 \cosh(2\eta)] \\
&\quad - K \cosh(4\eta) + 6\alpha^2 \cosh(4\eta)] \text{sech}(\eta)^6 \\
&\quad + \frac{\alpha^2 y^4}{32} [95\alpha^4 - 190\alpha^2 \gamma^2 + 95\gamma^4 - 9800\alpha^6 - 1920\alpha^4 \beta^2 + 9800\alpha^4 \gamma^2 \\
&\quad + 1249520a^8 + 86\alpha^4 \cosh(2\eta) - 172\alpha^2 \gamma^2 \cosh(2\eta) \\
&\quad + 86\gamma^4 \cosh(2\eta) - 1232\alpha^6 \cosh(2\eta) - 192\alpha^4 \beta^2 \cosh(2\eta) \\
&\quad + 1232\alpha^4 \gamma^2 \cosh(2\eta) - 1411744\alpha^8 \cosh(2\eta) \\
&\quad - 32\alpha^4 \cosh(4\eta) + 64\alpha^2 \gamma^2 \cosh(4\eta) - 32\gamma^4 \cosh(4\eta) \\
&\quad + 7616\alpha^6 \cosh(4\eta) + 1536\alpha^4 \beta^2 \cosh(4\eta) \\
&\quad - 7616\alpha^4 \gamma^2 \cosh(4\eta) + 233728\alpha^8 \cosh(4\eta) - 22\alpha^4 \cosh(6\eta) \\
&\quad + 44\alpha^2 \gamma^2 \cosh(6\eta) - 22\gamma^4 \cosh(6\eta) - 944\alpha^6 \cosh(6\eta) \\
&\quad - 192\alpha^4 \beta^2 \cosh(6\eta) + 944\alpha^4 \gamma^2 \cosh(6\eta) \\
&\quad - 8032\alpha^8 \cosh(6\eta) + \alpha^4 \cosh(8\eta) - 2\alpha^2 \gamma^2 \cosh(8\eta) \\
&\quad + \gamma^4 \cosh(8\eta) + 8\alpha^6 \cosh(8\eta) - 8\alpha^4 \gamma^2 \cosh(8\eta) \\
&\quad + 16\alpha^8 \cosh(8\eta)] \text{sech}(\eta)^{10} + 2\alpha^4 \beta y^3 [-9K - 210 \alpha^2 \\
&\quad - 8K \cosh(2\eta) + 144\alpha^2 \cosh(2\eta) + K \cosh(4\eta) \\
&\quad - 6\alpha^2 \cosh(4\eta)] \text{sech}(\eta)^6 \tanh(\eta) + \frac{\alpha^2 \beta y^5}{80} [-515\alpha^4 + 1030\alpha^2 \gamma^2 \\
&\quad - 515\alpha^4 + 60200\alpha^6 - 60200\alpha^4 \gamma^2 - 7215920\alpha^8 \\
&\quad - 596\alpha^4 \cosh(2\eta) + 1192\alpha^2 \gamma^2 \cosh(2\eta) - 596\gamma^4 \cosh(2\eta) \\
&\quad + 29792\alpha^6 \cosh(2\eta) - 29792\alpha^4 \gamma^2 \cosh(2\eta) \\
&\quad + 6533824\alpha^8 \cosh(2\eta) - 28\alpha^4 \cosh(4\eta) + 56\alpha^2 \gamma^2 \cosh(4\eta) \\
&\quad - 28\gamma^4 \cosh(4\eta) - 28448\alpha^6 \cosh(4\eta) + 28448 \alpha^4 \gamma^2 \cosh(4\eta) \\
&\quad - 749248\alpha^8 + (52\alpha^4 - 104\alpha^2 \gamma^2 + 52\gamma^4 + 1952\alpha^6 - 1952\alpha^4 \gamma^2}
\[ +16192\alpha^8 \cosh(6\eta) - (\alpha^4 - 2\alpha^2\gamma^2 + \gamma^4 + 8\alpha^6 - 8\alpha^4\gamma^2 \]
\[ +16\alpha^8 \cosh(6\eta) \sech(\eta)^{10} \tanh(\eta) \] (28)

and so on, where \( \eta = (\alpha x - \gamma t) \).

Therefore, The exact solution of (23) with the initial condition (24) and (25) in closed form will be

\[ U(x, y, t) = K - 6\alpha^2 \tanh(\alpha x + \beta y - \gamma t), \]
where \( K = \frac{1}{2} \frac{\alpha^4 + 8\alpha^4 - \alpha^2 - \beta^2}{\alpha^2 - \beta^2} \) and \( \alpha, \beta, \gamma \) are arbitrary constants.

### 4 Numerical results

Table 1

<table>
<thead>
<tr>
<th>( (x_i, y_i) )</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.2013610^{-8}</td>
<td>1.2013610^{-8}</td>
<td>6.1003510^{-8}</td>
<td>1.4865310^{-7}</td>
<td>3.8152310^{-7}</td>
</tr>
<tr>
<td>0.2</td>
<td>2.0013510^{-8}</td>
<td>2.0013510^{-8}</td>
<td>5.3018710^{-8}</td>
<td>5.0012310^{-8}</td>
<td>4.5032110^{-8}</td>
</tr>
<tr>
<td>0.3</td>
<td>1.9187610^{-8}</td>
<td>1.9187610^{-8}</td>
<td>1.3398610^{-7}</td>
<td>1.1296710^{-7}</td>
<td>1.5004310^{-7}</td>
</tr>
<tr>
<td>0.4</td>
<td>1.8700110^{-8}</td>
<td>1.8700110^{-8}</td>
<td>2.3129310^{-7}</td>
<td>2.0059110^{-7}</td>
<td>1.8397610^{-7}</td>
</tr>
<tr>
<td>0.5</td>
<td>1.6234110^{-8}</td>
<td>1.6234110^{-8}</td>
<td>3.6597110^{-7}</td>
<td>3.1201210^{-7}</td>
<td>2.7800910^{-7}</td>
</tr>
</tbody>
</table>

Continuing this process the complete solution \( u = \lim_{n \to \infty} \phi_n \) found by means of \( n \)-term approximation \( \phi_n = \sum_{k=0}^{n} \nu_k \). For this example, for-term approximation \( \phi_4 = \nu_0 + \nu_1 + \nu_2 + \nu_3 \) gives the best convergence.

The error, complete(analytical) and the numerical results are given and compared each other in table 1 for some values of \( x, y, \) and \( t = 0.5 \).

As expected, the numerical solution in table 1 is clearly indicated that how the decomposition scheme obtains efficient result much closer to the actual solutions. It is also worth noting that the advantage of the decomposition methodology shows a fast convergence of the solution. Clearly, the series solution methodology can be applied to much more complicated any other nonlinear problems as well.
Figure 1: An approximate numerical solution for $\phi_3$ with $t = 0.5$ when $\alpha = 0.1$, $\beta = 0.01$, and $\gamma = 1$.

Figure 2: The analytic solution for the (2+1)-dimensional Boussinesq equation with $t = 0.5$ when $\alpha = 0.1$, $\beta = 0.01$, and $\gamma = 1$. 
In this work, we successfully apply the homotopy perturbation method to approximate the solution of nonlinear (2+1)-dimensional Boussinesq equation. It is shown that HPM needs much less computational work compared with traditional method. It is apparently seen that this method is very powerful and an efficient technique for solving different kinds of problems arising in various fields of science and engineering and present a rapid convergence for the solution. Although goal of He's homotopy perturbation method was to find a technique to unify linear and nonlinear, ordinary or partial differential equation for solving initial and boundary value problems.

References


Received: March, 2011