Metric Dimension in Fuzzy Graphs –

A Novel Approach

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Abstract

In this paper we introduce a novel approach for finding the fuzzy metric dimension in fuzzy graphs. Definitions and theorems related to fuzzy metric dimension in fuzzy path and fuzzy cycle are presented. These concepts are illustrated through examples.

Keywords: Fuzzy graph, fuzzy path, fuzzy cycle, metric dimension, fuzzy metric

1 Introduction

Graphs are simply models of relations. A graph is a convenient way of representing information involving relationship between objects. The objects are represented by vertices and relations by edges. Graphs emerge naturally as a mathematical model of the observed real world system in applications such as transportation, routing, communication etc. Numerous algorithms have been developed to optimize the problems in graphs. In practice, due to the sheer number of optimization criteria they impose realistic environments which are often so complex that even the most sophisticated implementations become computationally unmanageable. There are many ambiguously formulated information or imprecisely quantified physical data values in real world applications. When there is vagueness in the description of the objects or in the relationships or on both, it is natural that we need to design a new model. For this purpose, fuzzy technology can be employed for making a decision to complex problems.
Fuzzy graphs [2] have recently raised increasing attention, especially in areas where understandable models are needed. Application of fuzzy graphs are widespread, especially in the field of clustering analysis, neural networks, pattern recognition, decision making and expert systems.

The first definition of a fuzzy graph was given by Kauffmann in 1973 [4], based on Zadeh’s fuzzy relations. But it was Rosenfeld who considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs in 1975. During the same time Yeh and Bang have also introduced various concepts in fuzzy graphs [2].

Nair and Chang [6] defined the concept of cycle and a fuzzy cycle in fuzzy graphs. Perchant and Bloch gave a generic definition of fuzzy morphism between graphs, which uses a pair of fuzzy relations, one on the vertices and the other on the edges [7].

A location problem [3] deals with the choice of a set of points (vertices) for establishing certain facilities in a given graphical model. It takes into account different criteria and verifies a given set of constraints that optimally fulfill the needs of the users. Location problems deal with a wide set of fields where it is usually assumed that exact data are known. However, in real applications, the location of the facility considered can be full of linguistic vagueness that can be approximately modeled using networks with fuzzy values. Thus fuzzy location problems on networks arise. Jose et.al., [3] considered the concepts of $\alpha$-cuts on these problems. But in our paper, we give a novel approach to find the locating set of the given graph using fuzzy metric. We call the locating set as fuzzy metric basis and the number of elements in fuzzy metric basis as fuzzy metric dimension.

This paper is structured as follows: In section 2, we give the necessary definitions that are required to study the forth coming section. Section 3 defines fuzzy metric dimension of fuzzy graphs. The fuzzy metric dimension of fuzzy path and fuzzy cycles are discussed in sections 4 and 5 respectively. The final section gives the conclusion.

2 Fuzzy Metric Dimension of Fuzzy Graph

As we concentrate on fuzzy graphs in which a vertex set is a fuzzy set, we view the edge set also as a fuzzy set induced by the fuzzy set on the vertex set. Hence we have the following definition of fuzzy graph.

A fuzzy graph $G$ is a 2-tuple $(V, E)$ where $V$ is a non empty set of vertices $\{v_1, v_2, \cdots, v_n\}$ and $E$ is the nonempty finite set of edges such that $\mu: V \rightarrow [0, 1]$ and $\sigma: V \times V \rightarrow [0, 1]$
where \( \sigma(v_i, v_j) = \begin{cases} \min(\mu(v_i), \mu(v_j)), & \text{for } i \neq j \\ 0, & \text{for } i = j \end{cases} \)

For any \( v \in V \), if \( \mu(v) > 0 \) then we call \( v \) as an active vertex. If \( \mu(v) = 0 \) then we call \( v \) as an inactive vertex. In this paper, we assume that all the vertices as active vertices.

**Notation:** \( e_{ij} \) is the edge connecting the vertices \( v_i \) and \( v_j \). The weight of the edge \( e_{ij} \) given by \( \sigma(v_i, v_j) \) and is denoted by \( w(e_{ij}) \).

A fuzzy path from a vertex \( v_i \) to a vertex \( v_j \) in a fuzzy graph is a sequence of distinct vertices and edges starting from \( v_i \) and ending at \( v_j \). This is denoted by \( p(v_i, v_j) = p \). If \( v_i \) and \( v_j \) coincide in a fuzzy path \( p \) then we call this sequence as a fuzzy cycle.

Let \( P_{ij} \) be the set of all fuzzy paths \( p \) from \( v_i \) to \( v_j \). For \( v_i, v_j \in V \), we define the fuzzy set \( \mu_{ij}: P_{ij} \rightarrow [0,1] \) by \( \mu_{ij}(p) = \min_{e \in p} (w(e)) \) where \( p \in P_{ij} \).

Here \( \mu_{ij}(p) \) is called the weight of the path \( p \). The fuzzy path \( p \in P_{ij} \) for which \( \mu_{ij}(p) \) is minimum, is called as a fuzzy shortest path (FSP) between \( v_i \) and \( v_j \). The weight of this FSP is denoted by \( d^*(v_i, v_j) \).

Thus, \( d^* \) can be viewed as a fuzzy set, \( d^*: V \times V \rightarrow [0,1] \) where \( d^*(v_i, v_j) = \min_{p \in P_{ij}} (\mu_{ij}(p)) \) and \( d^*(v_i, v_i) = 0 \).

**Theorem 1:** \( d^* \) is a metric.

The proof is obvious.

**Remark:** For any two fuzzy shortest path \( p \) and \( q \) between \( v_i \) and \( v_j \), we consider the path with lesser number of intermediate vertices.
The 3-tuple \((V, d^*, t)\) [1] is defined as \(d^*(v_i, v_j, t) = \frac{t}{t + d^*(v_i, v_j)}\) where \(t\) is the number of intermediate vertices in the shortest path from which \(d^*\) is calculated [1]. For convenience, we denote \(d^*(v_i, v_j, t)\) as \(d(v_i, v_j)\).

**Notation:** In this paper, we denote the number of intermediate vertices between \(v_i\) and \(v_j\) in FSP as \(N(v_i, v_j)\).

Let \(G = (V, E, \mu)\) be a fuzzy graph. Let \(\mathcal{M}\) be a subset of \(V\). \(\mathcal{M}\) is said to be a fuzzy metric basis of \(G\) if for every pair of vertices \(x, y \in V \setminus \mathcal{M}\), there exists a vertex \(w \in \mathcal{M}\) such that \(d^*(w, x) \neq d^*(w, y)\). The number of elements in \(\mathcal{M}\) is said to be fuzzy metric dimension (FMD) of \(G\) and is denoted by \(\tilde{\beta}(G)\). The elements in \(\mathcal{M}\) are called as source vertices.

In this paper, we have determined the fuzzy metric dimension of path and cycle.

### 3 Fuzzy Metric Dimension of Fuzzy path

In this section, we determine the fuzzy metric dimension of fuzzy path.

**Theorem 2:** If \(G\) is a path then \(\tilde{\beta}(G) = 1\).

**Proof:** Consider a path \(P_n\) with \(n\) vertices \(v_1, v_2, \ldots, v_i, \ldots, v_n\). Let \(v_1\) be the source vertex.

Here \(i < j\).

To prove: \(d(v_i, v_j) \neq d(v_1, v_j) \forall i, j\)

Suppose \(d(v_i, v_j) = d(v_1, v_j) \text{ \(\text{----------------(1)}\)}\)
As there is only one path joining $v_1$ and $v_i$ and the number of intermediate vertices ($t$) is $i - 2$.

$$d^*(v_1, v_i) = \frac{i - 2}{(i - 2) + d^*(v_1, v_i)}$$

Similarly

$$d^*(v_1, v_j) = \frac{j - 2}{(j - 2) + d^*(v_1, v_j)}$$

From equation (1),

$$\Rightarrow \frac{i - 2}{(i - 2) + d^*(v_1, v_i)} = \frac{j - 2}{(j - 2) + d^*(v_1, v_j)}$$

$$\Rightarrow (i - 2)(j - 2) + (i - 2) d^*(v_1, v_j) = (i - 2)(j - 2) + (j - 2) d^*(v_1, v_i)$$

This can be written as

$$\frac{i - 2}{i - 2} + \frac{j - i}{i - 2} = \frac{d^*(v_1, v_j)}{d^*(v_1, v_i)}$$

$$\Rightarrow 1 + \frac{j - i}{i - 2} = \frac{d^*(v_1, v_j)}{d^*(v_1, v_i)}$$

Now $d^*(v_1, v_i) \geq d^*(v_1, v_j)$ as $i < j$

$$\frac{d^*(v_1, v_j)}{d^*(v_1, v_i)} \geq 1$$

$$\frac{d^*(v_1, v_i)}{d^*(v_1, v_j)} < 1$$

$$1 + \frac{j - i}{i - 2} < 1$$

$$j - i < 0$$

$$j < i$$
This is a contradiction for $i < j$.

Hence the proof

**Corollary 1:** If $G$ is a path and $\bar{\beta}(G) = 1$ then $\bar{d}(v_1, v_i) < \bar{d}(v_1, v_j)$ for $i < j$.

**Theorem 3:** If $P_n$ is a path on $n$ vertices and $v_k$ is an intermediate vertex in $P_n$, $v_i$ and $v_j$ are two vertices on either side of $v_k$ then $\bar{d}(v_k, v_i) = \bar{d}(v_k, v_j)$ if and only if
\[
\frac{N(v_k, v_i)}{N(v_k, v_j)} = \frac{d^+(v_k, v_i)}{d^+(v_k, v_j)}
\]

**Proof:** Assume $\bar{d}(v_k, v_i) = \bar{d}(v_k, v_j)$
\[
\frac{k_1}{k_2 + d^+(v_k, v_i)} = \frac{k_2}{k_2 + d^+(v_k, v_j)}
\]
where $k_1 = N(v_k, v_i)$ and $k_2 = N(v_k, v_j)$.

Conversely, let
\[
\frac{N(v_k, v_i)}{N(v_k, v_j)} = \frac{d^+(v_k, v_i)}{d^+(v_k, v_j)}
\]

Corollary 2: Let $P_n$ be a path on $n$ vertices and $v_k$ is an intermediate vertex in $P_n$. If $v_i$ and $v_j$ are two vertices on either side of the vertex $v_k$
such that $N(v_k, v_i) = N(v_k, v_j)$ then $d^*(v_k, v_i) = d^*(v_k, v_j)$ if and only if $d^*(v_k, v_i) = d^*(v_k, v_j)$.

The proof is obvious.

**Note:** The ratio $\frac{N(v_k, v_i)}{N(v_k, v_j)} = d^*(v_k, v_i)$ in Theorem 3 is called fuzzy path index.

**Remark:** In the classical graph theory always the initial vertex is the source vertex for a path [4] whereas in fuzzy graph this need not be true, which is illustrated in the following example. Here, we show that, an intermediate vertex $v_3$ is a source vertex. See Figure 1.

![Figure 1: A fuzzy path with fuzzy weights in the edges](image)

Here, $d^*(v_3, v_4) = 0.2, d^*(v_3, v_2) = 0.2, d^*(v_3, v_4) = 0.1, d^*(v_3, v_5) = 0.1,$

$d^*(v_3, v_6) = 0.1$.

Clearly, $d^*(v_3, v_i) = d^*(v_3, v_j)$ for $i,j = 1,2,4,5,6$ and $i \neq j$.

### 4 Fuzzy Metric Dimension of Fuzzy Cycle

Let $C_n = (V, E, \mu_E)$ be a fuzzy cycle with $n$ vertices $v_1, v_2, \ldots, v_n, v_1$. Fix a vertex $v_1$, which we consider as a source vertex for cycle $C_n$. If $n$ is even, then $v_{\frac{n+1}{2}}$ is diametrically opposite vertex of $v_1$. If $n$ is odd, then $v_{\frac{n+1}{2}}$ and $v_{\frac{n+2}{2}}$ are the two diametrically opposite vertices of $v_1$.

If $n$ is even, let $P_1$ be the path $v_1, v_2, \ldots, v_{\frac{n+1}{2}}$ and $P_2$ be the path $v_1, v_n, v_{n-1} \ldots, v_{\frac{n+1}{2}}$.

If $n$ is odd, let $P_1$ be the path $v_1, v_2, \ldots, v_{\frac{n+1}{2}}$ and $P_2$ be the path $v_1, v_3, v_4, \ldots, v_{\frac{n+1}{2}}$.
Clearly, in either case $C_n = P_1 \cup P_2$. See Figure 2.

Case 1: Let $v_i$ and $v_j$ ($i < j$) be two vertices on $C_n$.

Both $v_i$ and $v_j \in P_1$ (or $P_2$).

Then both $v_i$ and $v_j$ will have the same FSP from $v_1$.

Case 2: $v_i \in P_1$ and $v_j \in P_2$ (or $v_j \in P_1$ and $v_i \in P_2$).

For each of $v_i$ and $v_j$ there are two paths from $v_1$ one through $P_1$ and the other through $P_2$.

If the FSP for $v_i$ is through $P_1$, then the FSP for $v_j$ is also through $P_1$, which is Case 1.

For $v_i$, if the FSP is through $P_2$ then the minimum value $w(e)$ may occur in any one of the edges between $v_1$ and $v_j$ or between $v_j$ and $v_i$. In the first case, the FSP for $v_j$ also occurs through $P_2$. Thus for both $v_i$ and $v_j$ the FSP is
through $P_2$, which is nothing but Case 1.

Suppose the minimum value $w(e)$ occurs in any one of the edges between $v_i$ and $v_j$, then the fuzzy shortest path for $v_i$ is through $P_2$ and the FSP for $v_j$ is through $P_1$.

**Lemma 1:** If $v_i, v_j \in C_n$ such that they have the same FSP from $v_1$ then $\bar{\beta}(C_n) = 1$.

**Proof:** If both $v_i$ and $v_j$ have the same FSP from $v_1$, then $v_1, v_i$ and $v_j$ will be in the same path. Then, proof follows from Theorem 2.

**Lemma 2:** Let $C_n$ be a cycle and $v_i \in P_1$ and $v_j \in P_2$ such that the FSP for $v_i$ is through $P_2$ and FSP for $v_j$ is through $P_1$ then $\tilde{d}(v_1, v_i) = \tilde{d}(v_1, v_j)$ if and only if $N(v_1, v_i) = N(v_1, v_j)$.

**Proof:** Let $N(v_1, v_i) = k_1$ and $N(v_1, v_j) = k_2$.

Assume $\tilde{d}(v_1, v_i) = \tilde{d}(v_1, v_j)$

$$k_1 = \frac{k_1}{k_1 + d^*(v_1, v_i)} = \frac{k_2}{k_2 + d^*(v_1, v_j)}$$

$k_1 d^*(v_2, v_j) = k_2 d^*(v_1, v_i)$

$k_1 = k_2$ as $d^*(v_1, v_i) = d^*(v_1, v_j)$.

i.e., $N(v_1, v_i) = N(v_1, v_j)$

Conversely, suppose $N(v_1, v_i) = N(v_1, v_j)$

Then clearly $\tilde{d}(v_1, v_i) = \tilde{d}(v_1, v_j)$ as $d^*(v_1, v_i) = d^*(v_1, v_j)$.

**Theorem 4:** If $C_n$ is a cycle then $\bar{\beta}(C_n) \leq 2$
**Case 1:** If the two vertices \( v_i \) and \( v_j \) belong to either \( P_1 \) or \( P_2 \).

By Lemma 1, we get \( \beta(C_n) = 1 \).

**Case 2:** Suppose \( P_2 \) is the FSP for \( v_i \) and \( P_1 \) is the FSP for \( v_j \). If \( N(v_i, v_j) = N(v_i, v_j) \) then by Lemma 2, \( \tilde{d}(v_i, v_j) = d(v_i, v_j) \). This implies that \( \beta(C_n) \neq 1 \).

Include \( v_2 \) or \( v_n \) as a source vertex so that \( N(v_k, v_i) \neq N(v_k, v_j) \) where \( k = 2 \) or \( n \).

\[ \Rightarrow \tilde{d}(v_k, v_i) \neq \tilde{d}(v_k, v_j). \]

Therefore \( M = \{v_1, v_k\} \) where \( k = 2 \) or \( n \).

Hence, \( \beta(C_n) = 2 \)

Combining Case 1 and 2, we get \( \beta(C_n) \leq 2 \).

**Remark:** Khuller et al. [5] proved that in classical graph theory \( \beta(C_n) = 2 \) for a cycle. But in fuzzy graph, we have proved that \( \beta(C_n) \leq 2 \).

**5 Conclusion**

In this paper, we have defined fuzzy graph and introduced a novel approach for defining FMD in fuzzy graph. We have proved that FMD for fuzzy path is one and obtained an upper bound for FMD in a fuzzy cycle.

**References**


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