Inference for Lomax Distribution under Generalized Order Statistics

Morteza Sadeghi Moghadam\textsuperscript{1}, Farhad Yaghmaei\textsuperscript{2} and Manoochehr Babanezhad\textsuperscript{2}* 

\textsuperscript{1}Department of Statistics, Faculty of Sciences 
Golestan University, Gorgan, Golestan, Iran 
\textsuperscript{2}Department of Statistics, Faculty of Sciences 
Golestan University, Gorgan, Golestan, Iran 

f_yaghmaei@yahoo.com, m.babanezhad@gu.ac.ir

Abstract

In this study, based on generalized order statistics the Bayesian and the classical estimations of the parameters, the reliability and the hazard functions of Lomax distribution are investigated. The Bayesian estimators are obtained through conjugate prior for the shape and the scale parameters of this distribution. This is done theoretically under the symmetric loss function and the asymmetric loss function. Finally, simulation study is carried out to compare different Bayesian estimators based on different loss functions with the classical estimators.

Mathematics Subject Classification: 62F31

Keywords: Generalized order statistics, Bayesian estimator, Asymmetric loss function, Lomax distribution

Corresponding author*: Manoochehr Babanezhad, e-mail: m.babanezhad@gu.ac.ir

1 Introduction

Generalized order statistics (gos) was introduced by Kamps (1995) as an unified distribution theoretical set-up which contains a variety of approaches of ordered random variables with different interpretations. Moreover, many other models of ordered random variables, e.g., ordinary order statistics, order statistics with nonintegral sample size, sequential order statistics, k-record values, and progressively type II censoring are particular cases of gos.
Random variables $X(r, n, \bar{m}, k), 1 \leq r \leq n$, are said to be gos if their joint density function is of the form

$$f(x_1, \ldots, x_n) = k \prod_{j=1}^{n-1} \gamma_j \left[ \prod_{i=1}^{n-1} F^{m_i}(x_i) f(x_i) \right] F^{-1}(x_n) f(x_n)$$

on the cone $F^{-1}(0) < x_1 < \cdots < x_n < F^{-1}(1)$ and $F(x) = 1 - F(x)$, where $F$ is an absolutely continuous distribution function with density function $f$, and $n \in \mathbb{N}$, $k \geq 1$, $\bar{m} = (m_1, \ldots, m_{n-1}) \in \mathbb{R}^{n-1}$ are the parameters such that $\gamma_r = k + n - r + \sum_{j=r}^{n-1} m_j \geq 1$ for all $r \in \{1, \ldots, n-1\}$, (for more details, see Kamps (1995)).

As discussed in Kamps (1995), for suitable choices of the parameters gos reduce to the well known ordered random variables, e.g., record values, progressively type II censoring and so on. If $m_1 = \cdots = m_{n-1} = -1$ and $k = 1$, then $X(r, n, \bar{m}, k)$ reduces to the record values and (1) gives the joint pdf of the $n$th upper record values. If $m_i = R_i$ for $i = 1, \ldots, m-1$, $m_i = 0$, for $i = m, \ldots, n-1$ and $k = R_m + 1$, then (1) gives the joint pdf of the progressively type II censoring samples from a sequence of independent identically distributed random variables. Some distributional properties of gos studied and minimum variance linear unbiased estimates of the parameters of exponential distribution obtained based on gos (See, Ahsanullah, 2000; Malinowska et al., 2006). Also, based on gos from Weibull distribution the approaches of Bayesian and non-Bayesian estimation are discussed by Aboeleneen (2010).

The Lomax distribution has an important position in the field of lifetime testing. This distribution is useful for modeling and analyzing the lifetime data in medical and biological sciences, engineering (Lomax, 1954; Habibullah and Ahsanullah, 2000). Abd Ellah (2006) compared the approaches of Bayesian and non-Bayesian estimation from Lomax distribution using record values (see also Howlader and Hossain, 2002). The distribution function of Lomax distribution is given by

$$F(x; \alpha, \beta) = 1 - (1 + \frac{x}{\beta})^{-\alpha}, \ x > 0, \ \alpha, \beta > 0,$$

where $\alpha$ and $\beta$ are the shape and the scale parameters, respectively. The reliability function $R(t)$, and the hazard function $H(t)$ at time $t$ for the Lomax distribution are respectively, given by

$$R(t) = (1 + \frac{t}{\beta})^{-\alpha}, \ t > 0, \ H(t) = \frac{\alpha}{\beta} (1 + \frac{t}{\beta})^{-1}, \ t > 0.$$

In life testing and reliability problems, the nature of losses are not always symmetric and hence the use of SELF is forbidden and unacceptable in many situations. Inappropriateness of SELF has also been pointed out by different
authors (Ferguson, 1967; Varian, 1975). This leads to the idea that an asymmetrical loss function may be more appropriate. One of the most popular of them, is LINEX loss function which was introduced by Varian (1975). Under the assumption that the minimal loss occurs at \( \hat{\phi} = (\hat{\alpha}, \hat{\beta}) \), the LINEX loss function for \( \phi = (\alpha, \beta) \) can be expressed as

\[
L(\Delta) \propto e^{c\Delta} - c\Delta - 1; \quad c \neq 0,
\]

where \( \Delta = (\hat{\phi} - \phi) \) and \( \hat{\phi} \) is an estimate of \( \phi \). The Bayesian estimator of \( \phi \), denoted by \( \hat{\phi}_{BL} \) under the LINEX loss function is given by

\[
\hat{\phi}_{BL} = -\frac{1}{c}\ln[E_{\phi}e^{-c\phi}].
\]

2 Bayesian estimators

Suppose that \( X(1, n, \bar{m}, k), X(2, n, \bar{m}, k), \cdots, X(n, n, \bar{m}, k), k \geq 1 \), are \( n \) gos based on the density function from the Lomax distribution. According to (1) and (2), the likelihood function is

\[
L(\alpha, \beta; x) = k(\frac{\alpha}{\beta})^n \prod_{j=1}^{n-1} \gamma_j e^{-\alpha u - v},
\]

where

\[
u = \sum_{i=1}^{n-1} \ln(1 + \frac{x_i}{\beta})^{m_i+1} + k \ln(1 + \frac{x_n}{\beta}), \quad v = \sum_{i=1}^{n} \ln(1 + \frac{x_i}{\beta}).
\]

2.1 Known scale parameter

In case where \( \beta \) is known, we assume a gamma \( (a, b) \) conjugate prior for \( \alpha \) as

\[
\pi(\alpha|a, b) = \frac{b^a}{\Gamma(a)}\alpha^{a-1}e^{-b\alpha}
\]

where the hyperparameters \( a > 0 \) and \( b > 0 \) are given. Combining the likelihood function (6) and the latter prior distribution, we obtain the posterior density of \( \alpha \) given the data as follows

\[
\pi(\alpha|X, \beta) = \frac{(b + u)^{n+a}}{\Gamma(n + a)}\alpha^{n+a-1}e^{-\alpha(u+b)},
\]

where \( u \) is defined in (7). The Bayesian estimators for \( \alpha \) under the SELF can then be obtained as

\[
\hat{\alpha}_{BS} = E(\alpha|X) = \frac{n + a}{b + u}.
\]
Similarly, the Bayesian estimator for the reliability function $R(t)$ with fixed $t > 0$ can be obtained as

$$\hat{R}_{BS}(t) = \left\{ 1 + \frac{\ln(1 + \frac{t}{\beta})}{b + u} \right\}^{-(n+a)} \quad ,$$

and for the hazard function $H(t)$ as

$$\hat{H}_{BS}(t) = \frac{n + a}{(b + u)(t + \beta)} .$$

(11)

Alternatively, under the LINEX loss function the Bayesian estimator of $\alpha$, is given by

$$\hat{\alpha}_{BL} = -\frac{1}{c} \ln \left[ \int_{0}^{\infty} e^{-ca} \pi(\alpha|\beta) d\alpha \right] .$$

(13)

Then, it follows from (9) that

$$\hat{\alpha}_{BL} = \frac{n + a}{c} \ln \left( 1 + \frac{c}{b + u} \right) .$$

(14)

Moreover, the Bayesian estimators for $R(t)$ and $H(t)$ under the LINEX loss function can be obtained as

$$\hat{R}_{BL}(t) = -\frac{1}{c} \ln \left[ \sum_{i=0}^{\infty} \frac{(-c)^i}{i!} \left\{ 1 + \frac{i \ln(1 + \frac{t}{\beta})}{b + u} \right\}^{-(n+a)} \right]$$

(15)

and

$$\hat{H}_{BL}(t) = \frac{n + a}{c} \ln \left( 1 + \frac{c}{(b + u)(t + \beta)} \right) .$$

(16)

2.2 Unknown scale and shape parameters $\beta$ and $\alpha$

In case where both the scale and the shape parameters $\beta$ and $\alpha$ are unknown, determining a general joint prior for $\alpha$ and $\beta$ lead to computational complexities. We use Soland’s method (1969) in order to solve this problem in which he considered a family of joint prior pdfs that places a continuous pdf on $\alpha$ and a discrete distribution on $\beta$.

We assume that the scale parameter $\beta$ is restricted to a finite number of values $\beta_1, \beta_2, \cdots, \beta_k$ with respective prior probabilities $\psi_1, \psi_2, \cdots, \psi_k$ such that $0 \leq \psi_j \leq 1$, and $\sum_{j=1}^{k} \psi_j = 1$ [i.e. $\Pr(\beta = \beta_j) = \psi_j$]. Further, suppose that conditional upon $\beta = \beta_j$, $\alpha$ has a natural conjugate prior with distribution gamma($a_j, b_j$) with density

$$\pi(\alpha|\beta = \beta_j) = \frac{b_j^{a_j}}{\Gamma(a_j)} \alpha^{a_j-1} e^{-b_j\alpha}, \quad a_j, b_j > 0.$$
Then given the set of $n$ Gos values $x$, the conditional posterior pdf of $\alpha$ is
\[
\pi(\alpha|\beta = \beta_j, x) = \frac{(b_j + u_j)^{n+a_j}}{\Gamma(n+a_j)} \alpha^{n+a_j-1} e^{-\alpha(b_j+u_j)}, \quad \alpha, a_j, b_j > 0, \quad (18)
\]
with $u_j = \sum_{i=1}^{n-1} \ln(1 + \frac{x_i}{\beta_j})^{m+1} + k \ln(1 + \frac{x_n}{\beta_j})$.

The marginal posterior mass function of $\beta_j$ can then be obtained as
\[
P_j = Pr(\beta = \beta_j|x) = A \int_0^\infty \psi_j \frac{b_j^{a_j}}{(a_j)\beta_j^a} \alpha^{n+a_j-1} e^{-\alpha(b_j+u_j)} d\alpha = A \frac{\psi_j e^{-v_j} b_j^{a_j} \Gamma(n+a_j)}{\beta_j^a (b_j+u_j)^{n+a_j} \Gamma(a_j)}. \quad (19)
\]
where $v_j = \sum_{i=1}^{n} \ln(1 + \frac{x_i}{\beta_j})$ and $A$ is the normalized constant given by
\[
A^{-1} = \sum_{j=1}^{k} \frac{\psi_j e^{-v_j} b_j^{a_j} \Gamma(n+a_j)}{\beta_j^a (b_j+u_j)^{n+a_j} \Gamma(a_j)}. \quad (20)
\]

Therefore, we can derive the Bayesian estimators of $\alpha$ and $\beta$ under the SELF using the posterior pdfs (18) and (19) as the following
\[
\hat{\alpha}_{BS} = \int_0^\infty \sum_{j=1}^{k} P_j \alpha \pi(\alpha|\beta = \beta_j, x) d\alpha = \sum_{j=1}^{k} \frac{P_j (n+a_j)}{(b_j+u_j)}, \quad (21)
\]
and
\[
\hat{\beta}_{BS} = \sum_{j=1}^{k} P_j \beta_j. \quad (22)
\]

Similarly, the Bayesian estimators of $R(t)$ and $H(t)$ with fixed $t > 0$ are given respectively by
\[
\hat{R}_{BS}(t) = \sum_{j=1}^{k} P_j \left(1 + \ln(1 + \frac{t}{\beta_j})\right)^{-(n+a_j)}, \quad (23)
\]
\[
\hat{H}_{BS}(t) = \sum_{j=1}^{k} P_j \frac{(n+a_j)}{(b_j+u_j)(t+\beta_j)}. \quad (24)
\]

Alternatively, under the LINEX loss function, the Bayesian estimator for $\alpha$ and $\beta$ can be derived as
\[
\hat{\alpha}_{BL} = -\frac{1}{c} \ln \left[ \int_0^\infty \sum_{j=1}^{k} P_j e^{-\alpha c} \pi(\alpha|\beta = \beta_j, x) d\alpha \right]
= -\frac{1}{c} \ln \left[ \sum_{j=1}^{k} P_j \left(1 + \frac{c}{b_j+u_j}\right)^{-(n+a_j)} \right], \quad (25)
\]
and
\[ \hat{\beta}_{BL} = -\frac{1}{c} \ln \left[ \sum_{j=1}^{k} P_j e^{-c\beta_j} \right]. \] (26)
respectively. Similarly, the Bayesian estimator for the reliability function \( R(t) \) and \( H(t) \) with fixed \( t > 0 \) can be respectively obtained as follows
\[ \hat{R}_{BL}(t) = -\frac{1}{c} \ln \left[ \sum_{j=1}^{k} \sum_{i=0}^{\infty} P_j \left( \frac{-c}{i!} \right)^i \left\{ 1 + \frac{i \ln(1 + \frac{t}{\beta_j})}{b_j + u_j} \right\}^{-(n+a_j)} \right], \] (27)
\[ \hat{H}_{BL}(t) = -\frac{1}{c} \ln \left[ \sum_{j=1}^{k} P_j \left\{ 1 + \frac{c}{(b_j + u_j)(t + \beta_j)} \right\}^{-(n+a_j)} \right]. \] (28)
Before the calculations, we should know the values of \((\beta_j, \psi_j)\) and the hyperparameters \((a_j, b_j)\) in the conjugate prior (17). It is not always possible to know the values of the hyperparameters \((a_j, b_j)\). So, for obtaining the values \((a_j, b_j)\), we use the expected value of the reliability function \( R(t) \) conditional on \( \beta = \beta_j \), which is given using (17) by
\[ E_{\alpha|\beta_j}[R(t)|\beta = \beta_j] = \int_{0}^{\infty} \exp \left\{ -\alpha \ln \left( 1 + \frac{t}{\beta_j} \right) \right\} \frac{b_j^{a_j} \alpha^{a_j-1}}{\Gamma(a_j)} e^{-ab_j} d\alpha \]
\[ = \left\{ 1 + \ln \left( 1 + \frac{t}{\beta_j} \right) \right\}^{-a_j}. \] (29)
Thus, for these two prior values \( R(t_1) \) and \( R(t_2) \), the values of \((a_j, b_j)\) can be obtained numerically from (29) for each value \( \beta_j \). If there are no prior beliefs, a nonparametric procedure can be used to estimate the corresponding two different values of \( R(t) \) (for more details, see Martz and Waller (1982), p. 105).

3 Maximum likelihood estimation

The likelihood function based on the first \( n \) gos for the Lomax distribution is given in (6). Hence the log likelihood function is as the following form,
\[ \ln L(\alpha, \beta; x) = \ln k + \sum_{j=1}^{n-1} \gamma_j + n \ln \alpha - n \ln \beta - \alpha u - v. \] (30)
where \( u \) and \( v \) are defined in (7).
When \( \beta \) is known, we obtain the MLE of \( \alpha \) as
\[ \hat{\alpha}_{ML} = \frac{n}{\sum_{i=1}^{n-1} \ln(1 + \frac{x_i}{\beta})^{m_i+1} + k \ln(1 + \frac{x_m}{\beta})}. \] (31)
The case where $\alpha$ and $\beta$ are both unknown, $\hat{\alpha}_{ML}$ and $\hat{\beta}_{ML}$, can be obtained by solving the likelihood equations. $\hat{\beta}_{ML}$ can then be obtained numerically from the equation

$$\frac{n}{\hat{\beta}_{ML}} \left[ \sum_{i=1}^{n-1} \frac{(m_i + 1)x_i}{(x_i + \hat{\beta}_{ML})} + kx_n \right] - \sum_{i=1}^{n} \frac{1}{(x_i + \hat{\beta}_{ML})} = 0. \quad (32)$$

The corresponding MLEs $\hat{R}_{ML}(t)$ and $\hat{H}_{ML}(t)$ of reliability function $R(t)$ and hazard function $H(t)$ are given respectively by replacing $\alpha$ and $\beta$ by $\hat{\alpha}_{ML}$ and $\hat{\beta}_{ML}$ in equation (3). Note that by choosing $m_i = -1, i = 1, \cdots, n - 1$ and $k = 1$, we obtain Bayesian and ML estimators corresponds to the $n$th upper record from Lomax distribution which coincides with that obtained by Abd Ellah (2006). Also, by considering special cases of progressively type II censored sample, we can obtained the estimators under type II censored sample and complete sample.

## 4 Simulation

To investigate the performance of the Bayesian and non-Bayesian estimators, we conducted a simulation experiment in the following subsections.

### 4.1 The case of known $\beta$

In this case, random samples of progressively type II censored and upper record are generated based on the algorithm presented in Aboeleneen (2010) as the following steps,

1. For given values $(a=3, b=2)$, we generate $\alpha = 2.32$ from the prior pdf (9).

2. Using the value $\alpha = 2.32$ from step 1, with known $\beta = 2$ and by choosing the parameters $m_i = R_i$ for $i = 1, \cdots, m-1$, $m_i = 0$, for $i = m, \cdots, n-1$ and $k = R_m + 1$ in the mentioned algorithm, for different choices of sample sizes and censoring schemes, we generate progressively Type II censored samples from the Lomax distribution.

3. Using the value $\alpha = 2.32$ from step 1, with known $\beta = 2$ and by choosing the parameters $m_1 = \cdots = m_{n-1} = -1$ and $k = 1$ in the mentioned algorithm, we generate $n$, $(n=3,5,7)$ upper record value from the Lomax pdf in (2).
4. The MLEs and the Bayesian estimators of $\alpha$, $R(t)$ and $H(t)$ (for $t = 2$) under SELF and LINEX loss function are computed by using the results in sections 2 and 3.

5. Step 1-4 are repeated 5000 times, and the mean squared error (MSE) for each method was calculated. The results are display in Tables 1 and 2 for different choices of the shape parameter of the LINEX loss function.

4.2 The case of unknown $\alpha$ and $\beta$

In the case where both parameters are unknown, specifying a general joint prior for $\alpha$ and $\beta$ leads to computational complexities, as it mentioned before. According to the mentioned algorithm and in line with Soland’s method a simulation study is done to compute the Bayesian and ML estimators under SELF and LINEX loss function, in the following steps,

1. We generate samples of upper record values with size $n$, ($n=3,5,7$) from the Lomax distribution ($\alpha = 3, \beta = 3$) with parameters $m_1 = \cdots = m_{n-1} = -1$ and $k = 1$. We approximate the prior for $\beta$ over the interval (2.5,3.4) by the discrete prior with $\beta$ taking the 10 values 2.5(0.1)3.4, each with probability 0.1.

2. We generate different progressively type II censored sample for different sample sizes from the Lomax pdf ($\alpha = 3, \beta = 3$), with parameters $m_i = R_i$ for $i = 1, \cdots, m-1$, $m_i = 0$, for $i = m, \cdots, n-1$ and $k = R_m + 1$.

3. According to the samples in step 1 and 2, we estimate two values of the reliability function using a nonparametric procedure which propose by Martz and Waller (1982).

4. Substituting the two prior values obtained in step 3 into (29), the hyperparameters $(a_j, b_j)$ are obtained numerically for given $\beta_j$, $j = 1, \cdots, 10$ using Newton-Raphson method.

5. The MLEs and Bayesian estimators of $\alpha, \beta, R(t)$ and $H(t)$ under SELF and LINEX loss function are computed using results in sections (2.2) and 3.

6. We repeated the above steps for 5000 times in order to evaluate the MSEs of each method. The results are shown in Tables 3, 4 and 5.

5 Discussion

In this paper, based on gos we consider the classical and Bayesian inference procedures to estimate the two unknown parameters as well as the reliability
and hazard functions for Lomax distribution. Here, the maximum likelihood estimators of the unknown parameters $\alpha$ and $\beta$ have been obtained. Also, Bayesian estimators are obtained using both symmetric (squared error) and asymmetric (LINEX) loss functions. A comparison is made between the considered estimators through a simulation study. The estimators are obtained separately for record values and progressively type II censoring that establish some well known results. Table 1 to 5 show that the Bayesian estimators based on progressively type II censoring and record values are superior to the MLEs. It is also clear that for both censoring samples and record values, the MSEs decrease as sample size increases for two cases of known scale parameter and unknown scale and shape parameters. Also, simulation results for the progressively type II censoring samples in Tables 2, 4 and 5 show that for a fixed sample size, as more units are censored the MSEs increase as expected.
Table 3: MSEs of the estimators of $\alpha$, $\beta$, $R(t)$ and $H(t)$ for record values when $t=2$

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$n$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$R(t)$</th>
<th>$H(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$c=-2$</td>
<td>$c=-0.5$</td>
<td>$c=2$</td>
<td>$c=-2$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>3</td>
<td>1.67656</td>
<td>0.76505</td>
<td>0.83325</td>
<td>0.82433</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.74302</td>
<td>0.55043</td>
<td>0.89654</td>
<td>0.73000</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>0.52243</td>
<td>0.33201</td>
<td>0.68543</td>
<td>0.45645</td>
</tr>
<tr>
<td>$\beta$</td>
<td>3</td>
<td>0.04543</td>
<td>0.03421</td>
<td>0.04435</td>
<td>0.04437</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.03433</td>
<td>0.02243</td>
<td>0.03965</td>
<td>0.02765</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>0.01838</td>
<td>0.01543</td>
<td>0.01432</td>
<td>0.01497</td>
</tr>
<tr>
<td>$R(t)$</td>
<td>3</td>
<td>2.43266</td>
<td>1.52405</td>
<td>0.96953</td>
<td>0.85655</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>1.86504</td>
<td>1.16548</td>
<td>0.75465</td>
<td>0.77445</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>0.88763</td>
<td>0.78650</td>
<td>0.76055</td>
<td>0.65021</td>
</tr>
<tr>
<td>$H(t)$</td>
<td>3</td>
<td>0.00445</td>
<td>0.00185</td>
<td>0.00113</td>
<td>0.00104</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.00314</td>
<td>0.00204</td>
<td>0.00214</td>
<td>0.00183</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>0.00086</td>
<td>0.00054</td>
<td>0.00104</td>
<td>0.00081</td>
</tr>
</tbody>
</table>

Table 4: MSEs of the estimators of $\alpha$ and $\beta$ for progressively type II censored samples when $t=2$ and $c=-0.5$

<table>
<thead>
<tr>
<th>$(n,m)$</th>
<th>$(R_1,\ldots,R_m)$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$c=-2$</td>
<td>$c=-0.5$</td>
<td>$c=2$</td>
<td>$c=-2$</td>
</tr>
<tr>
<td>$(10,5)$</td>
<td>(0,...,0.5)</td>
<td>0.08554</td>
<td>0.03445</td>
<td>0.05596</td>
<td>0.02043</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.05456</td>
<td>0.03245</td>
<td>0.04032</td>
<td>0.01765</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.05965</td>
<td>0.03354</td>
<td>0.04145</td>
<td>0.01865</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.04565</td>
<td>0.03104</td>
<td>0.03567</td>
<td>0.01758</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.04954</td>
<td>0.03114</td>
<td>0.03596</td>
<td>0.01778</td>
</tr>
<tr>
<td>$(10,10)$</td>
<td>(0,...,10)</td>
<td>0.02468</td>
<td>0.02043</td>
<td>0.02367</td>
<td>0.01096</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.02254</td>
<td>0.02011</td>
<td>0.02225</td>
<td>0.01096</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.02365</td>
<td>0.02109</td>
<td>0.02245</td>
<td>0.01145</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.01547</td>
<td>0.01469</td>
<td>0.01607</td>
<td>0.00957</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.01654</td>
<td>0.01597</td>
<td>0.01607</td>
<td>0.00979</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00856</td>
<td>0.00454</td>
<td>0.00563</td>
<td>0.00453</td>
</tr>
<tr>
<td>$(20,10)$</td>
<td>(0,...,0,10)</td>
<td>0.00645</td>
<td>0.00435</td>
<td>0.00535</td>
<td>0.00365</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00664</td>
<td>0.00435</td>
<td>0.00545</td>
<td>0.00363</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00345</td>
<td>0.00295</td>
<td>0.00318</td>
<td>0.00176</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00324</td>
<td>0.00278</td>
<td>0.00293</td>
<td>0.00132</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00245</td>
<td>0.00201</td>
<td>0.00221</td>
<td>0.00087</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00258</td>
<td>0.00225</td>
<td>0.00154</td>
<td>0.00073</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00143</td>
<td>0.00132</td>
<td>0.00105</td>
<td>0.00056</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00123</td>
<td>0.00120</td>
<td>0.00059</td>
<td>0.00051</td>
</tr>
<tr>
<td>$(50,25)$</td>
<td>(25,0,...,0)</td>
<td>0.00087</td>
<td>0.00019</td>
<td>0.00024</td>
<td>0.00019</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00086</td>
<td>0.00054</td>
<td>0.00066</td>
<td>0.00020</td>
</tr>
</tbody>
</table>

Table 5: MSEs of the estimators of $R(t)$ and $H(t)$ for progressively type II censored samples when $t=2$ and $c=-0.5$

<table>
<thead>
<tr>
<th>$(n,m)$</th>
<th>$(R_1,\ldots,R_m)$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$c=-2$</td>
<td>$c=-0.5$</td>
</tr>
<tr>
<td>$(10,5)$</td>
<td>(0,...,0.5)</td>
<td>0.00971</td>
<td>0.00786</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00868</td>
<td>0.00567</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00847</td>
<td>0.00598</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00645</td>
<td>0.00435</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00664</td>
<td>0.00435</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00345</td>
<td>0.00295</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00324</td>
<td>0.00278</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00245</td>
<td>0.00201</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00258</td>
<td>0.00225</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00143</td>
<td>0.00132</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00123</td>
<td>0.00120</td>
</tr>
<tr>
<td>$(20,10)$</td>
<td>(0,...,0,10)</td>
<td>0.00544</td>
<td>0.00431</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00540</td>
<td>0.00406</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00478</td>
<td>0.00453</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.0065</td>
<td>0.00287</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00363</td>
<td>0.00312</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00176</td>
<td>0.00145</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00132</td>
<td>0.00112</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00087</td>
<td>0.000112</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00073</td>
<td>0.00067</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.000143</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00056</td>
<td>0.00043</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00051</td>
<td>0.00035</td>
</tr>
<tr>
<td>$(50,25)$</td>
<td>(1,...,1)</td>
<td>0.00065</td>
<td>0.00028</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00020</td>
<td>0.00025</td>
</tr>
</tbody>
</table>
References


Received: May, 2012