

## Cost-Efficiency by Arash Method in DEA

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### Abstract

Arash Method (AM) was recently proposed to distinguish between efficient decision making units (DMUs) and technical efficient ones where there are no any prices, weights or other assumptions between inputs and outputs of DMUs. This paper illustrates that AM is also able to evaluate the cost-efficiency of DMUs. Indeed, this paper represents that there is no require using cost-efficiency model with possessing AM with a proposition and some examples.

**Mathematics Subject Classification:** 90

**Keywords:** Data envelopment analysis, Arash Method, Cost efficiency.

### 1. Introduction

A nonparametric method in operations research, data envelopment analysis (DEA), estimates the relative efficiency of a set of peer decision making units (DMUs). There are many DEA models such as linear and nonlinear programming problems to evaluate the performance of firms or organizations with many different purposes and multiple inputs and outputs. However, there may rarely be a complete model in DEA to measure several purposes of firms such as ranking inefficient and technical efficient DMUs at the same time or benchmark DMUs simultaneously. Moreover, Khezrimotlagh et al. [2] illustrated that a technical efficient DMU may neither be efficient nor be more efficient than some inefficient ones and therefore they proposed a new method in DEA, called Arash Method (AM), which is able to distinguish between efficient DMUs and technical efficient ones with simultaneously ranking all inefficient and technical efficient DMUs together. This paper demonstrates another capability of AM to calculate cost-efficiency (CF) of DMUs. In this paper, Section 2 is the background, the relation between AM and CF model is illustrated in Section 3 and the paper is concluded in Section 4.

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## 2. Background

Efficiency is the ability of a firm to obtain maximum (minimum) outputs (inputs) from a given set of inputs (outputs), whereas cost-efficiency requires achieving the lowest possible cost, given current prices and firm outputs. Allocation models, such as CF model, have been proposed to identify some types of inefficiency of firms when information on prices and costs are exactly available [1]. For instance, Figure 1 depicts the concept of cost efficiency where five DMUs labeled A, B, C, D and E with two inputs and a single constant output are evaluated and the available prices for input 1 and 2 are  $c_1$  and  $c_2$ . From the figure, E is a point in production possibility set (PPS) which produces the same amount of output, but with greater amounts of both inputs. Now, minimizing the linear combination of  $c_1x_1 + c_2x_2$  and using Farrell measure of radial efficiency [1], identifies the two points,  $E'$  and  $E''$ , and defines the “technical efficiency”, “allocative efficiency” and “cost efficiency” as following  $d(O, E')/d(O, E)$ ,  $d(O, E'')/d(O, E')$  and  $d(O, E'')/d(O, E)$ , respectively.

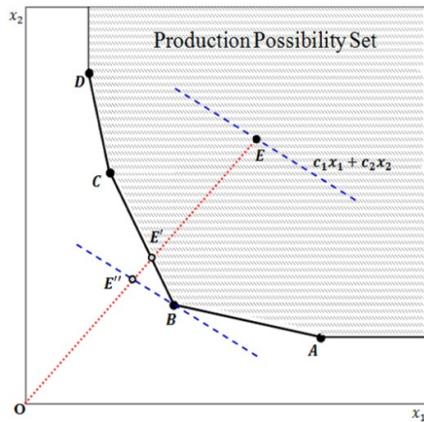


Figure 1: Cost Efficiency (CF)

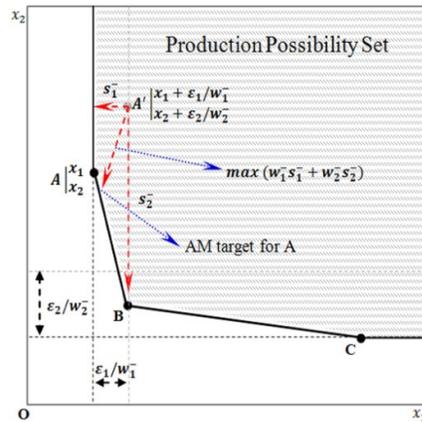


Figure 2: Arash Method (AM)

On the other hand, the conventional DEA models (where unit price and cost information are not available) are usually able to identify technical efficient DMUs, however, they are not usually able to distinguish between efficient DMUs and technical efficient ones in this case. Therefore, Khezrimotlagh et al. [2] proposed the Arash Method (AM) to characterize efficient DMUs among the technical efficient ones where the prices and weights are unknown. For instance, Figure 2 depicts the AM concept for three technical efficient DMUs labelled A, B and C, where an epsilon error occurs in the A’s inputs.

In order to illustrate the CF model and AM let us suppose that there are  $n$  DMUs ( $DMU_i, i = 1, 2, \dots, n$ ) with  $m$  nonnegative inputs ( $x_{ij}, j = 1, 2, \dots, m$ ) and  $p$  nonnegative outputs ( $y_{ik}, k = 1, 2, \dots, p$ ) for each DMU which at least one of its inputs and outputs are not zero. The  $\epsilon$ -AM and CF model are as following where  $DMU_l (l = 1, 2, \dots, n)$  is evaluated.

**$\epsilon$ -Arash Model:**

$$\begin{aligned} & \max \sum_{j=1}^m w_j^- s_j^- + \sum_{k=1}^p w_k^+ s_k^+, \\ & \text{Subject to} \\ & \sum_{i=1}^n \lambda_i x_{ij} + s_j^- = x_{lj} + \epsilon_j / w_j^-, \quad \forall j, \\ & \sum_{i=1}^n \lambda_i y_{ik} - s_k^+ = y_{lk}, \quad \forall k, \\ & \lambda_i \geq 0, \quad \forall i, \\ & s_j^- \geq 0, \quad \forall j, \\ & s_k^+ \geq 0, \quad \forall k. \end{aligned}$$

**Cost-Efficiency Model:**

$$\begin{aligned} & \min \sum_{j=1}^m c_j x_j, \\ & \text{Subject to} \\ & \sum_{i=1}^n \lambda_i x_{ij} \leq x_j, \quad \forall j, \\ & \sum_{i=1}^n \lambda_i y_{ik} \geq y_{lk}, \quad \forall k, \\ & \lambda_i \geq 0, \quad \forall i. \end{aligned}$$

In the above models,  $w_j^-$ , for  $j = 1, 2, \dots, m$ , and  $w_k^+$ , for  $k = 1, 2, \dots, p$ , allow the summations in the **AM** objective be meaningful and they can be the user specified weights obtained through values judgment, prices or cost information. In addition,  $c_j$ , for  $j = 1, 2, \dots, m$ , in CF model are the available common unit costs and prices. Moreover,  $x_j$ , for  $j = 1, 2, \dots, m$ , are variables for optimizing the inputs of an evaluated DMU in CF model and in  **$\epsilon$ -AM**,  $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_m)$ ,  $\epsilon_j \geq 0$ , for  $j = 1, 2, \dots, m$ , , which is usually considered as  $\epsilon = (\epsilon, \epsilon, \dots, \epsilon)$ , where  $0 \leq \epsilon < 1$ . Furthermore,  $s_j^-$ 's and  $s_k^+$ 's are nonnegative slacks, i.e., potential reducing of inputs and potential increasing of outputs, for  $j = 1, 2, \dots, m$  and  $k = 1, 2, \dots, p$ , respectively. The scores of  **$\epsilon$ -AM** and cost-efficiency model are also as following:

**$\epsilon$ -AM Score:**

$$A^* = \frac{\sum_{k=1}^p w_k^+ y_{lk} / \sum_{j=1}^m w_j^- x_{lj}}{\sum_{k=1}^p w_k^+ y_{lk}^* / \sum_{j=1}^m w_j^- x_{lj}^*},$$

where

$$\begin{cases} x_{lj}^* = x_{lj} + \epsilon_j / w_j^- - s_j^{-*}, \forall j, \\ y_{lk}^* = y_{lk} + s_k^{+*}, \forall k. \end{cases}$$

**CF Score:**

$$Cf^* = \frac{\sum_{j=1}^m c_j x_j^*}{\sum_{j=1}^m c_j x_{lj}^*},$$

where  $x_j^*$  is optimum of  $x_j$ ,  $\forall j$ .

### 3. Relations between AM and CF Model

Since in calculating the cost-efficiency of DMUs, it is focused to input values and their prices, therefore the potential increasing of slacks i.e.,  $s_k^+$ 's can be deleted in  **$\epsilon$ -AM** objective and score. Then, the  **$\epsilon$ -AM** objective is  $\max \sum_{j=1}^m w_j^- s_j^-$  and the  **$\epsilon$ -AM** score are as following

$$A^* = \frac{\sum_{j=1}^m w_j^- x_{lj}^*}{\sum_{j=1}^m w_j^- x_{lj}} \quad \& \quad \begin{cases} x_{lj}^* = x_{lj} + \epsilon_j / w_j^- - s_j^{-*}, \forall j, \\ y_{lk}^* = y_{lk}, \forall k. \end{cases}$$

The previous **AM** score in comparison with CF score suggests to define  $w_j^- = c_j$ , for  $j = 1, 2, \dots, m$ , in **AM**. Now, assume that  $\epsilon = 0$ . Then, from the score of **0-AM** it yields that,  $x_{lj}^* = x_{lj} - s_j^{-*}$ , for  $j = 1, 2, \dots, m$ . From the first

constraints of **0-AM** and CF model, let us define  $x_j = x_{lj} - s_j^-$ , for  $j = 1, 2, \dots, m$ . Therefore, it yields

$$\begin{aligned}
 c_j x_j &= c_j x_{lj} - c_j s_j^-, \text{ for } j = 1, 2, \dots, m, \\
 \Rightarrow \sum_{j=1}^m c_j x_j &= \sum_{j=1}^m (c_j x_{lj} - c_j s_j^-) \\
 \Rightarrow \min_x \sum_{j=1}^m c_j x_j &= \min_x \sum_{j=1}^m (c_j x_{lj} - c_j s_j^-) \\
 \Rightarrow \min_x \sum_{j=1}^m c_j x_j &= \sum_{j=1}^m c_j x_{lj} - \max_s \sum_{j=1}^m c_j s_j^- \\
 \Rightarrow \sum_{j=1}^m c_j x_j^* &= \sum_{j=1}^m c_j x_{lj} - \sum_{j=1}^m c_j s_j^{-*} \\
 \Rightarrow \sum_{j=1}^m c_j x_j^* &= \sum_{j=1}^m c_j (x_{lj} - s_j^{-*}) \\
 \Rightarrow \sum_{j=1}^m c_j x_j^* &= \sum_{j=1}^m c_j x_{lj}^* \\
 \Rightarrow \frac{\sum_{j=1}^m c_j x_j^*}{\sum_{j=1}^m c_j x_{lj}} &= \frac{\sum_{j=1}^m c_j x_{lj}^*}{\sum_{j=1}^m c_j x_{lj}} = A^*.
 \end{aligned}$$

On the other hand, the lower bound of  $x_j$  in the first constraints of CF is the linear combination of inputs, i.e.,  $\sum_{i=1}^n \lambda_i x_{ij}$ , for  $j = 1, 2, \dots, m$ , whereas the lower bound of  $-s_j^-$  in the first constraint of **0-AM** is  $\sum_{i=1}^n \lambda_i x_{ij} - x_{lj}$ . This shows that  $x_j$  in CF model only depends to the linear combination of  $x_{ij}$ 's and does not depend to  $x_{lj}$ , whereas  $s_j^-$  depends to both  $x_{lj}$  and the linear combinations of  $x_{ij}$ 's, therefore the definition of  $x_j = x_{lj} - s_j^-$  cannot be used for relationship between  $x_j$  and  $s_j^-$  in **0-AM**. However, selecting  $\varepsilon > 0$  bridges over the models and rectifies the dependences of  $s_j^-$  to  $x_{lj}$  in PPS. In other words, when the arbitrary  $\varepsilon > 0$  is considered, then the term " $x_{lj} + \varepsilon_j/w_j^-$ " allows that  $s_j^-$  to be optimum in the wider domain and it helps to bridge the gaps between CF model and **AM**. As a result, the definition of  $x_j = x_{lj} - s_j^-$  by selecting the diversity amounts for  $\varepsilon > 0$ , is meaningful to relation between  $x_j$  in CF model and  $s_j^-$  in  **$\varepsilon$ -AM**. Moreover, raising the amounts of  $\varepsilon > 0$  gives the closer results between the models.

Now, from the previous illustrations the following proposition and its corollary are proved.

**Proposition:** CF model is equivalence with  **$\varepsilon$ -AM** ( $\varepsilon > 0$ ), where the potential increase of outputs, i.e.,  $s_k^+$ 's, are eliminated in **AM** objective, and the score is as following

$$\begin{aligned}
 & \max \sum_{j=1}^m w_j^- s_j^-, & \text{Score:} \\
 & \text{Subject to} & A^* = \frac{1/\sum_{j=1}^m w_j^- x_{lj}}{1/\sum_{j=1}^m w_j^- x_{lj}^*}, \\
 & \sum_{i=1}^n \lambda_i x_{ij} + s_j^- = x_{lj} + \varepsilon_j/w_j^-, \forall j, & \text{where} \\
 & \sum_{i=1}^n \lambda_i y_{ik} - s_k^+ = y_{lk}, \forall k, & x_{lj}^* = x_{lj} + \frac{\varepsilon_j}{w_j^-} - s_j^-, \forall j. \\
 & \lambda_i \geq 0, \forall i, \\
 & s_j^- \geq 0, \forall j, \\
 & s_k^+ \geq 0, \forall k.
 \end{aligned}$$

**Corollary:** The  $\varepsilon$ -AM and CF model outcomes are the same when the amount of  $\varepsilon > 0$  is large enough.

In order to demonstrate the previous outcomes, consider three DMUs with two inputs and two outputs in Table 1. Assume that the prices of input1 and input2 are 4 and 2 for DMUs, respectively. Columns six and seven of Table 1 identify the results of applying 0-AM and CF model which are the same for all DMUs. In this example, DMU B has the best performer in comparison with DMUs A and C. Moreover, the technical efficiency scores of DMUs are depicted in column eight.

Table 1: Three DMUs with two inputs and one output.

DMUs	Input1	Input2	Output1	Output2	0-AM Score	CF Score	Tech. Ef.
A	3	2	3	3	0.3750	0.3750	0.9000
B	1	3	5	6	1.0000	1.0000	1.0000
C	4	6	6	6	0.4286	0.4286	0.6000

Now, let us suppose that the amount of output2 for DMU A is 6 instate of 3 in Table 1 and apply 0-AM and CF. According to Table 2, both DMUs A and B have the score of 1 by 0-AM and they are also technical efficient, whereas the CF model score is 0.6250 and 1 for DMUs A and B, respectively. This outcome clearly identifies the differences between 0-AM and technical efficiency in comparison with CF model and cost-efficiency, respectively.

Table 2: Example to depict the differences between 0-AM and CF model scores.

DMUs	Input1	Input2	Output1	Output2	0-AM	CF	Tech. Ef.
A	3	2	3	6	1.0000	0.6250	1.0000
B	1	3	5	6	1.0000	1.0000	1.0000
C	4	6	6	6	0.4286	0.4286	0.6000

Regarding to the previous proposition let us select the amount of epsilon such as  $\varepsilon = 0.01$ ,  $\varepsilon = 0.1$ ,  $\varepsilon = 1$ ,  $\varepsilon = 2$ , and  $\varepsilon = 4$ , and apply  $\varepsilon$ -AM. The results are represented in Table 3 which obviously depict the capabilities of  $\varepsilon$ -AM ( $\varepsilon > 0$ ) in comparison with current technical and mix DEA models. In fact, only 0.01 error is enough to characterize the frail performer of DMU A in comparison with DMU B. Furthermore, raising the amounts of epsilon gives the better domain to optimize the potential decrease of A’s inputs and identifies the recovered efficiency score for DMU A as it is demonstrated in Table 3.

**Table 3:** The  $\varepsilon$ -AM scores by changing  $\varepsilon > 0$ .

DMUs	<b>0.01-AM</b>	<b>0.1-AM</b>	<b>1-AM</b>	<b>2-AM</b>	<b>4-AM</b>
A	0.9981	0.9813	0.8125	0.6250	0.6250
B	1.0000	1.0000	1.0000	1.0000	1.0000
C	0.4286	0.4286	0.4286	0.4286	0.4286

Indeed,  $\varepsilon$ -AM ( $\varepsilon > 0$ ) examines that only a little error in each input of a DMU how much affects on its efficiency score within the production possibility set (PPS) and classifies its cost-efficiency amounts and the stabilities of results by increasing the amount of epsilon.

#### 4. Conclusion

In this paper, one of the capabilities of Arash Method (AM) to assess the performance evaluation of firms and organizations is introduced and it is proved that with possessing AM there is no require using the cost-efficiency model. The study focuses on the cost-efficiency model, however, AM is able to improve for calculating the revenue-efficiency and the profit-efficiency, too.

#### References

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