A Model of the Weighted Networks with the Preferential Growth of Weights

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Abstract

A model for the weighted networks is proposed. The model is based on the preferential growth of weights. By the mean-field theory, the distributions of the strength and weight for the network are provided and the analytical results show that each distribution has a power-law tail.

Keywords: Weighted networks, Strength distribution, Weight distribution, Power-law tail

1 Introduction

Many social, biological and technological systems can be well described by complex networks [1-3]. These systems share a common scale-free feature that the degree distribution $P(k)$ decays as a power law. Researchers have mainly focused on the topological property of the networks, that is, unweighted networks. However, many systems are best described by weighted networks, whose properties and dynamics depend not only on their structures but also on the connection weights between their nodes. For instance, the number of co-authored papers of two scientists is very important in the understanding of the web of scientists with collaborations [4], and the number of available seats in flights between two airports is an important quantity in the characterization of world wide airport networks (WAN).

A weighted network can be described by a weighted adjacency matrix $W$, whose element $w_{ij}$ represents the weight on the edge connecting node $i$ and...
For the sake of simplicity, we only consider undirected networks in this paper, where the weights are symmetric, i.e., \( w_{ij} = w_{ji} \). As a generalization of the degree, the strength \( s_i \) of node \( i \), defined as \( S_i = \sum_{j \in v(i)} w_{ij} \), where the sum runs over the set \( v(i) \) of neighbors of node \( i \), is an important quantity in weighted networks. The strength of a node integrates the information about its connectivity and the weights of its links. For instance, the strength in WAN provides the actual traffic going through a node and indicates the size and importance of an airport in a certain extent. For the scientific collaboration networks (SCN), the strength is a measure of scientific productivity. Recent studies [5,6,7] have shown that the distributions of the strength and weight are power-law tailed in many weighted networks. Many models have been proposed to investigate the mechanism responsible for the properties found in many real-world weighted networks.

In 2004, Barrat, Barthélemy, and Vespignani (BBV) proposed a model for the growth of weighted networks [8,9]. The model takes into account the coupled evolution in time of topology and weights. (i) Growth. Starting from an initial seed of \( N_0 \) vertices connected by links with assigned weight. At each time step, a new vertex \( n \) is added with \( m \) edges that are randomly attached to a previously existing vertex \( i \) according to the probability distribution

\[
\prod_{n \rightarrow i} = \frac{s_i}{\sum_j s_j}
\]

The weight of each new edge is fixed to a value \( w_0 \). (ii) Weights’ dynamics. The presence of the new edge \((n,i)\) will introduce variations of the existing weights across the network. In particular, we consider the local rearrangements of weights between \( i \) and its neighbors \( j \in v(i) \) according to the simple rule

\[
w_{ij}(t+1) = w_{ij}(t) + \Delta w_{ij}(t), \Delta w_{ij}(t) = \frac{\delta w_{ij}(t)}{s_i(t)}
\]

The BBV model suggests two ingredients of self-organization of weighted networks, strength preferential attachment and weight evolving dynamics.

In present paper, we introduce a model of weighted network based on the preferential growth of weight. At each time step, a new node is attached at random to a previously existing vertex \( i \), and the additional increase \( p s_i \) in the strength of node \( i \) is distributed among its all links, according to the rule : \( w_{ij}(t+1) = (1+p)w_{ij}(t) \). The analytical results show that the model can evolve a network with the power-law distributions of the strength and weight.

2 The model

Inspired by BBV model, we propose a model to study the self-organization of weighted evolving networks. The algorithm goes as follows:
(i) Initial condition: Starting with a small number ($m_0$) of connected nodes. The initial weight of each link is assigned 1.

(ii) Growth: Adding a new node that is attached at random to a previously existing vertex $i$.

(iii) Weight evolution: The additional increase $ps_i$ in the strength of the node $i$ selected in the previous step is distributed among all links, according to the rule $w_{ij}(t + 1) = (1 + p)w_{ij}(t)$, where the parameter $p(> 0)$ represents the innovative ability of nodes.

(iv) The whole process is repeated from step (ii), until the desired size of the network is reached.

3 Strength and weight distributions

We will investigate the strength and weight distributions by the analysis. The network’s evolution can be inspected analytically by studying the time evolution of the average value of $s_i(t)$ of node $i$ at time $t$, and by relying on the continuous approximation that treats $s$, and the time $t$ as continuous variables [10]. When a new edge is added to the network, the strength $s_i$ of vertex $i$ can increase either if the edge connects directly to $i$ or to one of its neighbor.

The evolution equation for $s_i$ is thus given by

$$\frac{ds_i}{dt} = \frac{1}{N}(1 + ps_i) + \sum_{j \in v(i)} \frac{1}{N}pw_{ij} = \frac{1 + 2ps_i}{N}$$

(1)

Where the system size

$$N(t) = m_0 + t \approx t$$

Then, we obtain

$$\frac{ds_i}{dt} = \frac{1 + 2ps_i}{t}$$

(2)

With the initial condition: $s_i(t_i) = 1$, where node $i$ was added to the system at time $t_i$.

The solution of Eq. (2) has the form

$$s_i(t) = (1 + 1)\left(\frac{t}{t_i}\right)^{2p} - \frac{1}{2p}$$

(3)

We can obtain from (3):

$$P(s_i(t) < s) = P(t_i > t(1 + \frac{1}{2p})\frac{1}{2p}(s + \frac{1}{2p})^{-\frac{1}{2p}})$$

(4)
Assuming that we add the nodes at equal time intervals to the system, i.e., \( t_i \) follows the uniform distribution over interval \((0, m_0 + t)\). Hence,

\[
P(s_i(t) < s) = 1 - \frac{t}{m_0 + t}(1 + \frac{1}{2p})^\frac{1}{2p}(s + \frac{1}{2p})^{-\frac{1}{2p}}
\]

\[
P(s) = \frac{\partial P(s_i(t) < s)}{\partial s} = \frac{t}{m_0 + t} \frac{1}{2p}(1 + \frac{1}{2p})^\frac{1}{2p}(s + \frac{1}{2p})^{-\frac{1}{2p}-1}
\]

\[
\to \frac{1}{2p}(1 + \frac{1}{2p})^\frac{1}{2p}(s + \frac{1}{2p})^{-\frac{1}{2p}-1}(t \to \infty)
\] (5)

The strength distribution of the network follows a power law for large \( s \), with the exponent \( r_s \):

\[
r_s = 1 + \frac{1}{2p} > 1
\] (6)

Similarly, it is possible to obtain analytical expressions for the evolution of weights and the relative statistical distribution. The weight \( w_{ij} \) increases by the addition of a new link either on \( i \) or on \( j \), and the corresponding equation can be written as

\[
\frac{dw_{ij}}{dt} = \frac{1}{N}pw_{ij} + \frac{1}{N}pw_{ij} = \frac{2pw_{ij}}{N}
\] (7)

With the initial condition \( w_{ij}(t_{ij}) = 1 \), where \( t_{ij} = \max(i, j) \) is the time at which the edge is established, The equation (7) can be solved as

\[
w_{ij}(t) = \left(\frac{t}{t_{ij}}\right)^{2p}
\] (8)

In the same way, we can obtain from (8),

\[
P(w_{ij}(t) < w) = P(t_{ij} > tw^{-\frac{1}{2p}})
\]

\[
= 1 - \frac{t}{m_0 + t}w^{-\frac{1}{2p}}
\] (9)

\[
P(w) = \frac{\partial P(w_{ij}(t) < w)}{\partial w} = \frac{t}{m_0 + t} \frac{1}{2p}w^{-\frac{1}{2p}-1}
\]

\[
\to \frac{1}{2p}w^{-\frac{1}{2p}-1}(t \to \infty)
\] (10)

Consequently, the weight distribution \( P(w) \) has also a power-law tail, i.e.,

\[
P(w) \propto w^{-r_w}, r_w = 1 + \frac{1}{2p}
\] (12)
4 Conclusions and discussions

In present paper, we have proposed a model of weighted network based on the preferential growth of weights. At each time step, the probability that each node is chosen is identical, but large $s_i$ yields larger increases $ps_i$ when $i$ is chosen for the addition of a new edge, therefore achieving a faster strength growth as time goes, and the weight has the similar effect. The model results in scale-free behavior for the strength and weight distributions, and the exponents are controlled by a parameter $p$.

The strength and weight distributions have the same power exponent, therefore there are similarly sensitive to the parameter $p$ and evolve from a delta function for $p=0$ (no evolution of the weights) to a very broad power law as $p \to \infty$. The precise microscopic dynamics ruling the network’s growth and the rearrangement of weights is therefore very relevant to the final distributions of strength and weight. However, because adding a new node that is attached at random to a previously existing vertex, the degree distribution of our model is not power-law tailed.

References


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