A Simple Economic Model about the Teamwork Pedagogy

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Abstract

This note applies the construct of the elasticity of substitution in economics to an analysis of the teaching effectiveness in using the team approach in classrooms. We present examples as based on the well-known constant-elasticity-of-substitution (CES) production function, which we further specialize into the particular cases of zero, one, and infinite elasticities of substitution between any two members of a team of three students. We construct a "pair-wise CES" production function of three inputs, which can easily be extended into \( n \) inputs. We highlight the important role of the elasticity of substitution in any production and thus shed light incidentally back onto the state of the economy.

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1 Introduction

In many fields of studies the use of team projects in classroom assignments is either encouraged or considered as a simple routine. Even in the field of mathematics known for its abstractness, educators in recent years have advocated such a pedagogy. One typical argument in favor of this approach of teaching and learning is that of developing students’ abilities in teamwork so that in their later work places they will function cooperatively and productively with their co-workers. To be sure, the setup of committees to engage in group decisions is a ubiquitous phenomenon; clearly there has to be some underlying premise for this, and it appears to be that collective wisdom and work is superior to that of an individual. We, as classroom instructors however, are always
G. L. Light

concerned about the problem of substitutability among group members in an assigned task; i.e., Student 2 does less work because Student 1 does more.

We will apply the idea of the well-known "CES (constant elasticity of substitution) production function" in economics to a modeling of students’ teamwork (for a similar treatment, cf. [6]). Students will be regarded as the production inputs and their common score from a project will be treated as the output.

Section 2 below will first introduce our model as based on the CES production function and then present three illustrative examples showing the significant effect of input substitutability on the production output. Section 3 will conclude with a summary remark.

2 The Model

Consider the following "pair-wise CES" production function \( y = f(x_1, x_2, x_3) \) of three inputs (cf. [10] for "bilateral elasticities of substitution" among multiple inputs, and [4] for the prevalent use of the CES production function in economics):

\[
y : \quad = y_{12} + y_{13} + y_{23}, \quad \text{where} \quad y_{12} = A_{\{1,2\}} \cdot (a_1 x_1^{\rho_{12}} + a_2 x_2^{\rho_{12}})^{(s/\rho_{12})}, \\
y_{13} = A_{\{1,3\}} \cdot (a_1 x_1^{\rho_{13}} + a_3 x_3^{\rho_{13}})^{(s/\rho_{13})}, \\
y_{23} = A_{\{2,3\}} \cdot (a_2 x_2^{\rho_{23}} + a_3 x_3^{\rho_{23}})^{(s/\rho_{23})},
\]

\( A_{\{i,j\}} \) \( \equiv \) an interaction coefficient between \( i \) and \( j \) in the production of \( y \), \( a_k \equiv \) a contribution factor by \( x_k \) in the production of \( y \), and \( \rho_{ij} \in (-\infty, 1] - \{0\} \) is a factor related to the substitutability between \( i \) and \( j \) in the production of \( y \), to be explained later, and \( s > 0 \) is the "elasticity of scale" measuring the percentage increase in \( y \) due to a one-percent increase in all the inputs.

We now display the following standard textbook results:

1. the "marginal product in \( y_{ij} \) of \( x_i \) or \( x_j \),"

\[
\frac{\partial y_{ij}}{\partial x_i} = s \left( \frac{y_{ij}}{x_i} \right) \left( \frac{a_i x_i^{\rho_{ij}}}{a_i x_i^{\rho_{ij}} + a_j x_j^{\rho_{ij}}} \right), \quad \text{and} \quad \frac{\partial y_{ij}}{\partial x_j} = s \left( \frac{y_{ij}}{x_j} \right) \left( \frac{a_j x_j^{\rho_{ij}}}{a_i x_i^{\rho_{ij}} + a_j x_j^{\rho_{ij}}} \right);
\]

2. the "pair-wise rate of technical substitution \( RTS_{(j/i)} \)" in \( 0 = dy_{ij} = \)
\[
\left( \frac{\partial y_{ij}}{\partial x_i} \right) dx_i + \left( \frac{\partial y_{ij}}{\partial x_j} \right) dx_j,
\]

\[
RTS_{(j/i)} : = \frac{dx_j}{dx_i} \equiv - \left( a_j \right) \left( \frac{x_j}{x_i} \right)^{1-\rho_{ij}} \equiv - \left( a_j \right) r_{(j/i)}^{1-\rho_{ij}} \quad (\text{notation: } r_{(j/i)} \equiv \frac{x_j}{x_i}),
\]

so that

\[
\frac{dRTS_{(j/i)}}{dr_{(j/i)}} = - \left( a_j \right) (1 - \rho_{ij}) r_{(j/i)}^{-\rho_{ij}},
\]

(3) the “pairwise elasticity of substitution between (i, j)” as defined by setting \( A_{ik} = A_{jk} = 0 \) (cf. e.g., \([1,9]\), for the significance of the construct of elasticity of substitution in economics),

\[
\dot{\sigma}_{(j/i)} := \frac{dr_{(j/i)}}{dRTS_{(j/i)}} = \frac{1}{1 - \rho_{ij}} \in [0, \infty) - \{1\};
\]

(4) the “output y elasticity of input \( x_i \)”

\[
x_i \frac{\partial y}{y} \frac{\partial x_i}{\partial x_i} = s \left( \frac{y_{ij}}{y} \right) \left( \frac{a_i x_i^{\rho_{ij}}}{a_i x_i^{\rho_{ij}} + a_j x_j^{\rho_{ij}}} \right) + s \left( \frac{y_{ik}}{y} \right) \left( \frac{a_i x_i^{\rho_{ik}}}{a_i x_i^{\rho_{ik}} + a_k x_k^{\rho_{ik}}} \right),
\]

so that \( \sum_{i=1}^{3} x_i \frac{\partial y}{\partial x_i} = s \equiv \text{the elasticity of scale, as expected.} \)
Example 1 For the above pair-wise CES production function (Equation (1)), set
\[
A_{\{1,2\}} = A_{\{1,3\}} = A_{\{2,3\}} = 10, \\
a_1 = 0.6, \ a_2 = 0.3, \ a_3 = 0.1, \\
\rho_{12} = \rho_{13} = \rho_{23} = 0.5, \text{ so that } \hat{\sigma}_{ij} = 2, \forall i, j, \text{ and} \\
s = 1. \\
\text{(11)} \quad \text{(12)} \quad \text{(13)} \quad \text{(14)}
\]
Then we have
\[
y = 10 \left[ (0.6\sqrt{x_1} + 0.3\sqrt{x_2})^2 + (0.6\sqrt{x_1} + 0.1\sqrt{x_3})^2 + (0.3\sqrt{x_2} + 0.1\sqrt{x_3})^2 \right].
\text{(15)}
\]
Suppose that
\[
x_1 = 9, \ x_2 = 4, \text{ and } x_3 = 1.
\text{(16)}
\]
Then we have
\[
y = 98.6, \\
\frac{x_1}{y} \frac{\partial y}{\partial x_1} = 0.785, \\
\frac{x_2}{y} \frac{\partial y}{\partial x_2} = 0.189, \text{ and} \\
\frac{x_3}{y} \frac{\partial y}{\partial x_3} = 0.026, \text{ so that} \\
\sum_{i=1}^{3} \frac{x_i}{y} \frac{\partial y}{\partial x_i} = 1 = s, \text{ as expected.}
\text{(17)} \quad \text{(18)} \quad \text{(19)} \quad \text{(20)} \quad \text{(21)}
\]
One may interpret the example here as Student 1, 2, and 3 respectively spent 9, 4, and 1 hours in a team project and they obtained a (common) score of 98.6 (which would be distributed in accordance with economic theory as $98.6 \times (0.785 + 0.189 + 0.026)$ however).

Example 2 Consider the same pair-wise CES production function (Equation (1)) and set
\[
A_{\{1,2\}} = A_{\{1,3\}} = A_{\{2,3\}} = 10, \\
a_1 = 0.75, \ a_2 = 0.25, \ a_3 = 0.1, \\
\rho_{13} = \rho_{23} = -\infty, \text{ so that } \hat{\sigma}_{(3/1)} = \hat{\sigma}_{(3/2)} = 0, \\
\rho_{12} = 0 \text{ so that } \hat{\sigma}_{(2/1)} \text{ is yet to be defined below,} \\
s = 1, \ x_1 = 81, \ x_2 = 16, \text{ and } x_3 = \epsilon \gtrsim 0.
\text{(22)} \quad \text{(23)} \quad \text{(24)} \quad \text{(25)} \quad \text{(26)}
\]
Then
\[
y_{13} = \lim_{\rho_{13} \to -\infty} 10 \left( 0.75 \times 81^{\rho_{13}} + 0.1 \epsilon^{\rho_{13}} \right)^{(1/\rho_{13})}
\]
\[
= 10 \lim_{\rho_{13} \to -\infty} \left( 0.1 \epsilon^{\rho_{13}} \right)^{(1/\rho_{13})} \left( 7.5 \left( \frac{81}{\epsilon} \right)^{\rho_{13}} + 1 \right)
\]
\[
= 10 \epsilon \geq 0.
\]

Similarly,
\[
y_{23} = 10 \epsilon \geq 0.
\]

It then follows that \( y := y_{12} + y_{13} + y_{23} \geq y_{12} \), and
\[
y_{12} \equiv \lim_{\rho_{12} \to 0} 10 \left( 0.75x_{1}^{\rho_{12}} + 0.25x_{2}^{\rho_{12}} \right)^{(1/\rho_{12})}, \text{ or} \]
\[
o = \lim_{\rho_{12} \to 0} \frac{\left( \frac{y_{12}}{10} \right)^{\rho_{12}} - (0.75x_{1}^{\rho_{12}} + 0.25x_{2}^{\rho_{12}})}{\rho_{12}} \quad \text{(cf. [11], p. 17)}
\]
\[
= \ln \left( \frac{y_{12}}{10} \right) - (0.75 \ln x_{1} + 0.25 \ln x_{2}), \text{ i.e.,}
\]
\[
y \gtrsim y_{12} = 10x_{1}^{0.75}x_{2}^{0.25}
\]
\[
= 10 \times 81^{0.75} \times 16^{0.25} = 540.
\]

Here we may imagine that the three students got \( 540/1000 = 54\% \) out of a term project, for which Student 1, 2, and 3 spent respectively 81, 16, and nearly 0 hours; yet since \( \hat{\sigma}_{(3/1)} = \hat{\sigma}_{(3/2)} = 0 \), neither Student 1 nor Student 2 could substitute for Student 3’s input \( x_{3} \) in the production of \( y \), which greatly lowered their joint score. In passing, we note that the above well-known “Cobb-Douglas” production function (Equation (32)) has
\[
\hat{\sigma}_{(2/1)} = \frac{1}{1 - \rho_{12}} = 1.
\]

Furthermore, consider the following constrained optimization (for the differential geometry involved in a constrained optimization and the topic of surface regularities, cf. e.g., [2, 3, 5, 7]):
\[
\text{Min } w_{1}x_{1} + w_{2}x_{2}, \quad \text{s.t. } 10x_{1}^{0.75}x_{2}^{0.25} = y_{12}^{*},
\]
where \( (w_{1}, w_{2}, y_{12}^{*}) \) are the given parameters with \( w_{i} \equiv \text{the cost of } x_{i} \) and \( y_{12}^{*} \equiv \text{a pre-set output quantity. In our present context, Student 1 and 2 have put respectively their time costs of } w_{1} \text{ and } w_{2} \text{ to be spent in doing the project, and they together seek to minimize their total time cost incurred for the work but} \)
subject to the attainment of their desired score \( y_{12}^* \). Then we have the following first-order conditions:

\[
\frac{w_2}{w_1} = \frac{\partial y_{12}/\partial x_2}{\partial y_{12}/\partial x_1} = \frac{2.5 \left( \frac{w_2}{x_2} \right)^{0.75}}{7.5 \left( \frac{w_1}{x_1} \right)^{0.25}} = \frac{1}{3} \left( \frac{x_1}{x_2} \right), \quad \text{and} \quad (36)
\]

\[
y_{12}^* = 10x_1^{0.75}x_2^{0.25} = 10 \left( \frac{3x_2w_2}{w_1} \right)^{0.75}x_2^{0.25} = 10 \left( \frac{3w_2}{w_1} \right)^{0.75}x_2; \quad \text{i.e., the critical input} \quad (37)
\]

\[
x_2^* = \left( \frac{y_0}{10} \right) \left( \frac{3w_2}{w_1} \right)^{-0.75}, \quad \text{and} \quad (38)
\]

\[
x_1^* = \left( \frac{3w_2}{w_1} \right) \left( \frac{y_{12}^*}{10} \right) \left( \frac{3w_2}{w_1} \right)^{-0.75} = \left( \frac{y_{12}^*}{10} \right) \left( \frac{3w_2}{w_1} \right)^{0.25}. \quad (39)
\]

Assuming that \( w_1 = w_2 \) and \( y_{12}^* = 950 \), then their optimal input hours are

\[
x_1^* = 125.027 \text{ and } x_2^* = \frac{1}{3}x_1^* = 41.676 \text{ (Equation (36)),} \quad (40)
\]

where the unequal optimal input quantities are attributed to the fact that Student 1 has a higher output elasticity of \( a_1 = 0.75 \) than that of Student 2 of \( a_2 = 0.25 \). That is, a 1% increase in \( x_1 \) is to result in a 0.75% increase in \( y_{12} \).

**Example 3** Finally we illustrate the case of \( \hat{\sigma}_{(2/1)} = \frac{1}{1-\rho_{12}} = \infty \) (for the significance of a zero or infinite elasticity of substitution, see, e.g., [8]) by setting \( \rho_{12} = 1 \) but otherwise keeping all the same given information as in the preceding Example 2. Then we have

\[
y \gtrsim y_{12} \equiv 10(0.75x_1 + 0.25x_2) = 7.5x_1 + 2.5x_2 = 7.5 \times 81 + 2.5 \times 16 = 647.5. \quad (41)
\]

Here it is interesting to note that due to the perfect substitutability between \( x_1 \) and \( x_2 \), to obtain the same score of \( y_{12} = 950 \) Student 1 alone can achieve the outcome by spending \( x_1^* = 950 / 7.5 \approx 127 \) hours in the project, compared with the previous \( x_1^* \approx 125 \) in combination with \( x_2^* \approx 42 \) from the Cobb-Douglas technology in the preceding Example 2.
3 Summary Remark

In this paper, we have modeled teamwork or group decision-making by a pairwise CES production function from economics. Although our illustration here used only three inputs (students), one could easily extend our model into \( n \) inputs by a sum of \( \binom{n}{2} \) terms of the same form of \( A_{ij} \left( a_i x_i^{\rho_{ij}} + a_j x_j^{\rho_{ij}} \right)^{(s/\rho_{ij})} \). One may also vary any of the parameters to derive additional insights into the matter. As mentioned in the beginning, our work here contributes not only to classroom pedagogy but also to beyond, since the simple fact is that input substitutability in output is a common universal concern. In particular, if we identify ”Student 1, 2, and 3” in our above examples with capital, labor, and natural resources, then we have shown that a diminishing non-renewable natural resource that is not substitutable is to lower the overall production of the economy and that a large substitutability between capital and labor may result in large unemployment. Returning to educational methodology, we recommend instructors design team projects that have high level of complementarity.

References


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