Few Problems on Infinitesimal Variations
of Tangential Submanifolds
of Para-Sasakian Manifolds

Indiwar Singh Chauhan

Department of Mathematics
Ganjdundwara (P.G.) College
Ganjdundwara, Kanshiram Nagar, U.P., India
indiwarsinghchauhan@ymail.com

Abstract

The purpose of this paper is to delineate the theory of infinitesimal variations of tangential submanifolds of Para-Sasakian Manifold. In the last, we have defined tangential submanifolds and obtained several new results.

Keywords: infinitesimal transformation, infinitesimal variation, r-paracontact, killing vector field, tangential submanifold.

1. INTRODUCTION

Let $M^n$ be an n-dimensional differentiable manifold in which there are given a tensor field $\phi$ of rank two, a vector field $\xi$, a 1-form $\eta$ and riemannian metric $g$ satisfying [2]

(1.1) $\eta^i \xi^j = \delta^i_j$

(1.2) $\eta^i = g_{ij} \xi^j$

(1.3) $\eta^i \phi^j = 0$

(1.4) $\phi^i \xi^j = 0$

(1.5) $g_{ij} \xi^j = \eta^i$

(1.6) $\phi^i = g_{ik} \phi^k$

(1.7) $\phi^i = \nabla^i \eta^j$

(1.8) $\phi^i = g_{ij} \xi^j$

(1.9) $g_{ij} \xi^j = \eta^i$

(1.10) $g_{ih} \phi^h \phi^i = g_{ik} - \Sigma_{\alpha} \eta^j \eta^k$

(1.11) $\nabla^i \phi^h = - \Sigma_{\alpha} \eta^h \delta^i_j + \Sigma_{\beta} \eta^h \xi^j \eta^i \alpha - (g_{ik} - \Sigma_{\alpha} \eta^j \eta^i \alpha) \Sigma_{\beta} \xi^j \beta$
then \((\phi, \xi, \eta, \alpha, \mathbf{g})\) is called an almost r-paracontact Riemannian structure and the manifold \(M^n\) is called almost r-paracontact Riemannian manifold of P-Sasakian manifold.

Let \(M^{2m+1}\) be a \((2m+1)\)-dimensional differentiable manifold of class \(C^\infty\) covered by a system of coordinate. The indices \(i, j, k, \ldots\) run over the range \(1, 2, 3, \ldots, (2m+1)\).

Let \(M^{2m+1}\) be \((2m+1)\)-dimensional differentiable manifold in which there are given a tensor field \(F^i_j\) of the form \((1,1)\), a vector field \(f^i\) and a 1-form \(\alpha^i\) satisfying [6]

\[
(1.12) \quad F^i_j F^j_k = - \delta^i_k + \alpha_k f^i \\
(1.13) \quad F^i_j f^j = 0 \\
(1.14) \quad \alpha_i F^i_j = 0 \\
(1.15) \quad \alpha_i f^i = 1 \\
(1.16) \quad g_{ij} f^i f^j = 1
\]

then \((F, f, \alpha)\) is called an Almost Contact Structure and the manifold \(M^{2m+1}\) is called an Almost Contact Manifold.

**Definition 1.1 : Almost Contact Metric Manifold :**

If there is a Riemannian metric \(g_{ij}\) in \(M^{2m+1}\) such that

\[
(1.17) \quad g_{is} F^i_j F^s_j = g_{ij} - \alpha_i \alpha_j \\
(1.18) \quad \alpha_j = f^i g_{ij}
\]

then the manifold \(M^{2m+1}\) is called an Almost Contact Metric Manifold.

It is easy to verify that, in a almost contact metric manifold

\[
(1.19) \quad F^i_j = F^i_j g_{ki} \text{ is skew-symmetric.}
\]

In a P-Sasakian type manifold, we have

\[
(1.20) \quad F^i_j = \nabla_j f^i \\
(1.21) \quad \nabla_k F^i_j = - g_{kj} f^i + \delta^i_k f^j
\]

wherein \(\nabla_k\) is the operator of covariant differentiation with regard to the Christoffel symbols formed with Riemannian metric \(g_{ij}\) and

\[
(1.22) \quad f_j = f^i g_{ij} \\
(1.23) \quad f_{bc} = f^i_b g_{ac} \\
(1.24) \quad f^{bc} = f^i_a g^{ac} \\
(1.25) \quad f_{by} = f^i_b g_{xy} \\
(1.26) \quad f^{ib} = f^i_a g^{ab}
\]

If \(F^i_j\) is anti-symmetric and the following condition

\[
(1.27) \quad F^i_j = \nabla_i f_j \text{ is satisfied, then } f^i \text{ is called a unit killing vector field.}
\]

From equations (1.20) and (1.21), we get

\[
(1.28) \quad \nabla_k \nabla_j f^i = - g_{kj} f^i + \delta^i_k f^j .
\]

**2. SUBMANIFOLDS OF A P-SASAKIAN MANIFOLD**

Let \(M^n\) be an n-dimensional Riemannian Manifold isometrically immersed in \(M^{2m+1}\) by \(x^i = x^i(y^a)\) the immersion \(i : M^n \rightarrow M^{2m+1}\), where the indices \(a, b, c, \ldots\) run over the range \(1, 2, 3, \ldots, n\).

If we take

\[
(2.1) \quad B^i_a = \partial_a x^i \text{ wherein } \partial_a = \partial / \partial y^a
\]
then $B^i_a$ are $n$ linearly independent vectors of $M^{2m+1}$ tangent to $M^n$.

Since the immersion is isometric then we have

\[(2.2) \quad g_{ab} = g_{ij} B^i_a B^j_b \]

Wherein $g_{ab}$ denotes the Riemannian metric of $M^n$ and

\[(2.3) \quad B^i_a B^j_b = B^j_i.\]

Let $(2m-n+1)$ mutually orthogonal unit normals to $M^n$ be denoted by $C^i_x$, where $x, y, z, \ldots$ run over the range $(n+1, n+2, \ldots, 2m+1)$.

We have

\[(2.4) \quad g_{ij} B^i_a C^j_x = 0\]

and the Riemannian metric tensor $g_{xy}$ of the normal bundle of manifold $M^n$ is given by

\[(2.5) \quad g_{xy} = g_{ik} C^i_x C^k_y = \delta_{xy}\]

Wherein

\[(2.6) \quad C^i_x C^k_y = C^k_x.\]

The equation of Gauss for $M^n$ is given by

\[(2.7) \quad \nabla_b B^i_a = h^i_{ba} C^i_x\]

The equation of Weingarten for $M^n$ is given by

\[(2.8) \quad \nabla_b C^i_x = - h^i_{ba} B^i_a\]

Wherein $\nabla_b$ denotes the operator of Vander Warden-Bortolotti covariant differentiation with regard to $g_{ba}$ and $g_{yx}$ along $M^n$, $h^i_{ba}$ and $h^i_{bx}$ are the second fundamental tensors of $M^n$ with regard to $C^i_x$ related by

\[(2.9) \quad h^i_{bx} = h^i_{bc} g^{ca} g_{yx}\]

\[(2.10) \quad h_{bx} = h^a_{cx} g_{ab}\]

and

\[(2.11) \quad g^{xy} h_{bx} = h^y_{cb}\]

Where $g^{xy}$ is the contravariant components of $g_{bc}$.

In the P-Sasakian Manifold, we have

\[(2.12) \quad F^i_j B^i_a = B^i_b f^b_a - C^i_x f^a_x\]

\[(2.13) \quad F^i_k C^k_x = B^i_a f^a_x + C^i_y f^x_y\]

\[(2.14) \quad f^i = B^i_a f^a + C^i_x f^x\]

\[(2.15) \quad f^i_c f^i + f^a_x f^a = 0\]

\[(2.16) \quad f^i_e f^i - f^i_y f^y = 0\]

\[(2.17) \quad \nabla_{bf} f^a = f^a_b + h^a_{bx} f^x\]

\[(2.18) \quad \nabla_{bf} f^a = - f^a_b - h^a_{bc} f^c\]

\[(2.19) \quad \nabla_{cf} f^a = - g_{bc} f^c + \delta^c_a f_b + f^a_x h^x_{bc} - h^a_{cx} f^i_b\]

\[(2.20) \quad \nabla_{cf} f^a = g_{ac} f^c - h^a_{xy} f^y + h^a_{bc} f^b_{ab}\]

Then we have [3]

\[(2.21) \quad \xi^i_a = \xi^a_{x} C^i_x\]
\(\phi^i_j B^i_a = F^b_a B^i_b\)  
(2.23) \(\phi^i_k C^k_x = F^x_a C^i_y\)

Where \(F^b_a\) is the tensor field of the type \((1,1)\) on \(M^n\) and \(F^x_a\) is the tensor field of the type \((1,1)\) on the normal vector bundle of \(M^n\) satisfying

\(\phi^i_k C^k_x = F^x_a C^i_y\)  
(2.26) \(\phi^i_k \xi^\alpha = 0\) 
(2.27) \(\eta^i_x F^x_y = 0\)

Wherein

\(\eta^i_x = \eta^i_x C^i_x\) 
(2.28) \(\eta^i_x = \eta^i_x C^i_x\)

In the P-Sasakian manifold, we have

(2.30) \(F^b_a h^x_{ac} - F^y_x h^y_{bc} + g_{bc} \sum \rho_{\alpha}^x \beta = 0\)

(2.31) \(F^b_a F^d_c h_{acx} = h_{bdx}\)

(2.32) \(F^a b h^c_{ax} = F^a c h^b_{bx}\)

(2.33) \(\phi^y_x = g_{x \alpha} \xi^\alpha = \phi^y_x\)

Multiplying equation (2.14) by \(\eta^i_x\) and using equations (2.28) as well as (2.29), we obtain

(2.34) \(\eta^i_j f^i = \eta^i_x f^x\)

Contracting equation (2.14) by \(g_{ji}\) and using the equations (1.18) as well as (1.22), we obtain

(2.35) \(\alpha_j = f_j = g_{ji} B^i_a f^a + g_{ji} C^i_x f^x\)

Multiplying equation (2.22) by \(f^a\) and using the equation (1.13), we get

(2.36) \(\phi^i_j B^i_a f^a = 0\)

Inserting equation (1.8) into equation (2.36), we get

(2.37) \(\phi^a_k B^k_a f^a = 0\)

Multiplying equation (2.36) by \(g_{ik}\) and using the equation (1.6), we get

(2.38) \(\phi^a_k B^k_a f^a = 0\)

Inserting equation (1.7) into equation (2.38), we obtain

(2.39) \(\phi^a_k B^k_a f^a = 0\)

By virtue of equations (1.6), (1.19), (2.12) and (2.22), we get

(2.40) \(\phi^a_b B^k_a = B^k_x f^x_b - C^k_x f^x_b\)

Multiplying equation (2.40) by \(g^{bd}\) and using equations (1.6), (1.23) and (1.25), we obtain

(2.41) \(\phi^{a kd} B^k_a = B^k_x f^{x d} - C^k_x f^{x d}\)

Multiplying equation (2.40) by \(\eta^a j\) and using equations (2.28) and (2.29), we get
Problems on infinitesimal variations

(2.42) \( \eta^a \xi_b = 0. \)

3. INFINITESIMAL VARIATIONS OF SUBMANIFOLDS OF P-SASAKIAN MANIFOLD

We consider an infinitesimal variation of \( M^n \) in \( M^{2m+1} \) given by

\[
3.1 \quad x^{*i} = x^i + \mu^i(y) \varepsilon
\]

Where \( \varepsilon \) is an infinitesimal and \( \mu^i(y) \) is a vector field of \( M^{2m+1} \) defined along the submanifold \( M^n \).

Differentiating equation (3.1) partially on both sides, we get

\[
3.2 \quad \partial_a x^{*i} = \partial_a x^i + (\partial_a \mu^i) \varepsilon
\]

Inserting the relation \( \partial_a x^i = B^i_a \) in equation (3.2), we get

\[
3.3 \quad B^*^i_a = B^i_a + (\nabla_a \mu^i) \varepsilon
\]

If we take

\[
3.4 \quad \delta B^i_a = B^*^i_a - B^i_a
\]

and neglecting terms of higher order than one with respect to \( \varepsilon \), we get

\[
3.5 \quad \delta B^i_a = (\nabla_a \mu^i) \varepsilon
\]

If we put

\[
3.6 \quad \mu^i = B^i_b \mu^b + C^i_a \mu^x
\]

Differentiating equation (3.6) covariantly, we get

\[
3.7 \quad \nabla_a \mu^i = \nabla_a (B^i_b \mu^b + C^i_a \mu^x)
\]

By virtue of equations (2.7), (2.8) and (3.7), we obtain

\[
3.8 \quad \nabla_a \mu^i = (\nabla_a B^i_b - h^{ia} \mu^x) B^1_b + (\nabla_a \mu^x + h^{ia} \mu^b) C^i_x
\]

Multiplying equation (3.8) by \( \eta^a \xi \) and using equations (2.28) and (2.29), we get

\[
3.9 \quad \eta^a \xi \nabla_a \mu^i = (\nabla_a \mu^x + h^{ia} \mu^b) \eta^a \xi
\]

Multiplying equation (3.5) by \( \eta^a \xi \) and using equation (3.9), we have

\[
3.10 \quad \eta^a \xi \delta B^i_a = (\nabla_a \mu^x + h^{ia} \mu^b) \eta^a \xi \varepsilon
\]

Remark 3.1:

If the vector field \( \mu^i \) becomes the unit killing vector field \( \xi^i \). For this, we consider an infinitesimal variation

\[
3.11 \quad x^{*i} = x^i + \xi^i \varepsilon
\]

of a submanifold \( M^n \) of a P-Sasakian manifold \( M^{2m+1} \) which carries a point \( x^i(y) \) of the submanifold to a point \( x^{*i}(y) \) given by equation (3.11) of the varied submanifold, \( \xi^i \) being a unit killing vector field of \( M^{2m+1} \) defined along the submanifold \( M^n \) and \( \varepsilon \) is an infinitesimal.
Differentiating equation (3.11) partially, we get
\[ \partial_a x^i = \partial_a x^i + (\partial_a t^i)\epsilon \]
Using the relation \( \partial_a x^i = B_a^i \) in equation (3.12), we get
\[ B_a^i = B_a^i + (\nabla_a t^i)\epsilon \]
If we put
\[ \delta B_a^i = B_a^{*i} - B_a^i \]
and neglecting terms of higher order than one with regard to \( \epsilon \), we get
\[ \delta B_a^i = (\nabla_a t^i)\epsilon \]
Equation (2.14) can be written as
\[ t^i = B_b^i + C_b^i f^a \]
Differentiating equation (3.16) covariantly yield
\[ \nabla_a t^i = \nabla_a (B_b^i + C_b^i f^a) \]
By virtue of equations (2.7), (2.8) and (3.17), we get
\[ \nabla_a t^i = (\nabla_a B_b^i - h_b^a f^a)B_b^i + (\nabla_a f^i + h_b^a f^b)C^i_x \]
As a consequence of equations (2.15), (2.16), (2.17), (2.18) and (3.18), we get
\[ \nabla_a t^i = f_b^i B_b^i - f_a^i C^i_x \]
Inserting the value of \( \nabla_a t^i \) in the equation (3.15), we obtain
\[ \delta B_a^i = (f_b^i B_b^i - f_a^i C^i_x)\epsilon \]
Multiplying equation (3.19) by \( \eta_a^i \) and using equations (2.28) and (2.29), we obtain
\[ \eta_a^i \nabla_a t^i = - \eta_a^i f_a^i \]
From equation (2.42) and equation (3.21), we get
\[ \eta_a^i \nabla_a t^i = 0 \]
Multiplying equation (3.15) by \( \eta_a^i \) and using equation (3.21), we obtain
\[ \eta_a^i \delta B_b^i = - \eta_a^i f_a^i \epsilon \]
As a consequence of equation (2.42) and equation (3.23) provides
\[ \eta_a^i \delta B_b^i = 0 \]
Now we consider infinitesimal variations of unit normals \( C^i_y \). We denote by \( C^{**i}_y \), \( 2m-n+1 \) mutually orthogonal unit normals to the varied submanifold and by \( C^{**i}_y \), the vectors obtained from \( C^i_y \) by the parallel displacement from the varied point \( x^*y^i \) to the original point \( (x^i) \). Then we have
\[ C^{**i}_y = C^i_y + \Gamma^i_{jk}(x^i + \mu \epsilon)\mu^j C^k_y \]
Wherein \( \Gamma^i_{jk} \) denotes the Christoffel symbols formed with \( g_{ij} \).
Now we put
\[ \delta C_y^i = C^{**i}_y - C^i_y \]
and assume that \( \delta C_y^i \) is of the form
\[ \delta C_y^i = (B_a^i x^a_y + C^i_x x^y)\epsilon \]
Putting the value of $\delta C^i_y$ and $C^{**i}_y$ in the equation (3.26) and neglecting terms of higher order than one with respect to $\varepsilon$, we obtain

$$C^{**i}_y = C^i_y - \Gamma^i_{jk} \mu^j C^k_y \varepsilon + (B^i_a \tau^a_y + C^x \tau^x_y) \varepsilon$$

Applying operator $\delta$ to the equation (2.4), using the relation $\delta g_{ik} = 0$ and equations (3.5), (3.27), we get

$$g_{ik}(V^i_a \mu^a) C^k_x = \delta g_{ab} B^i_a \tau^b_x + g_{ij}(V^j_a \mu^a + h^i_{ac} \mu^c) C^k_x + g_{ab} \tau^b_x = 0$$

By virtue of equations (2.4), (2.5) and (3.34), we get

$$g_{ij}(V^j_a \mu^a + h^i_{ac} \mu^c) C^k_x + g_{ab} \tau^b_x = 0$$

Theorem 3.1:

To show that $\tau$ is skew-symmetric in the submanifolds of a P-Sasakian Manifolds.

Proof:

Applying the operator $\delta$ to the equation (2.5) and using the relation $\delta g_{ij} = 0$ and equation (3.27), we obtain

$$g_{ik}(V^i_a \mu^a C^k_x + \delta g_{ab} B^i_a \tau^b_x + C^x \tau^x_y) = 0$$

By virtue of equations (2.4), (2.5) and (3.34), we get

$$\tau^a_x g_{xy} + \tau^a_y g_{xz} = 0$$

Using the relation $\tau^a_x g_{xy} = \tau_{xy}$, the equation (3.35) yields

$$\tau_{xy} + \tau_{yx} = 0$$

i.e.

$$\tau_{xy} = - \tau_{yx}$$

Hence $\tau$ is skew-symmetric with respect to indices $x$ and $y$.

Multiplying equation (3.33) by $g_{ac}$ and using the relations $g_{ac} \tau^a_x = \tau_{bx}$, $\nabla_b = g_{ab} \nabla^a$ and the equation (2.10), we obtain

$$\tau^a_x = (V^a_{cx} + h^a_{cb} \mu^b)$$

Multiplying equation (3.38) by $g^xy$ and using the relation $g^{xy} h_{cbx} = h^y_{cb}$ and the equation (3.37), we get

$$\tau^a_x = V^a_{cx} + h^a_{ab} \mu^b$$

Now we shall obtain the variations of $f^a_b$, $f^a_b$, $f^a_y$, $f^a_y$, $f^a$ and $f^a$, we put

$$\tau^a_x = V^a_{cx} + h^a_{ab} \mu^b$$

for $\delta f^a_b$ and $\delta f^a_b$. 

\[ (x + \mu \varepsilon) B^a_b = B^a_b (f^a + \delta f^a) - C^a_b (f^b + \delta f^b) \]
Comparing tangential and normal parts on both sides, we get equation (3.51), we obtain
\[ (3.49) \delta f_x \]
and
\[ (3.47) \delta b_i \]
equation (3.46), we obtain
\[ \mu (3.41) \{\]
Using the equations (1.21), (2.12), (2.13), (2.14), (3.33) and (3.39) in the equation (3.41), we obtain
\[ (3.44) \delta f_b = \{(f_b \mu^y - \mu f_b^y\} + (V_b \mu^x - h_b x \mu^x) f_b + \tau^x_b f_b + \tau^x_b f_b^y \} \]
and
\[ (3.43) \delta f_b = \{(f_b \mu^x - \mu f_b^x\} + (V_b \mu^x - h_b x \mu^x) f_b + \tau^x_b f_b + \tau^x_b f_b^x \} \]
for \( \delta f_y \) and \( \delta f_b \).

Inserting the equations (3.3) and (3.28) into the equation (3.45), we obtain
\[ \mu (3.46) \{\]
Using the equations (1.21), (2.12), (2.13), (2.14), (3.33) and (3.39) in the equation (3.46), we obtain
\[ (3.47) \delta f_y = \{(f_y \mu^x - \mu f_y^x\} - (V_c \mu^x - h_c x \mu^x) f_y + \tau^y_b f_y - f_b^y f_y = f_b^y f_y \} \]
and
\[ (3.48) \delta f_y = \{(f_y \mu^x - \mu f_y^x\} - (V_c \mu^x - h_c x \mu^x) f_y + \tau^y_b f_y - f_b^y f_y = f_b^y f_y \} \]
for \( \delta f_y \) and \( \delta f_x \).

Inserting the equations (3.3) and (3.28) into the equation (3.50), we get
\[ \mu (3.51) \{\]
Using the equations (1.21), (2.12), (2.13), (2.14), (3.33) and (3.39) in the equation (3.51), we obtain
\[ (3.52) \delta f_b = \{(f_b \mu^x - \mu f_b^x\} - (V_c \mu^x - h_c x \mu^x) f_b + \tau^x_z f_b = f_b^x f_b \} \]
and
\[ (3.53) \delta f_b = \{(f_b \mu^x + f_b^y \mu^y\} - (V_c \mu^x - h_c x \mu^x) f_b - \tau^y f_b \} \]
for \( \delta f_b \) and \( \delta f_x \).
By virtue of equations (2.2), (2.4), (3.6) and (3.57), we get

\[ \delta t = \{(f^3_j \mu^x - f^3_b \mu^b) - \tau^x_j f^3_j\} \varepsilon \]

With the help of equation (3.3), we have

\[ \psi_j(x+\mu\varepsilon)B_{s}\mu^a = \psi_j B^a \mu^b + \mu^k (\nabla \psi_j) B_{s}^a \mu^e + \mu^k (\nabla \psi_j) (\nabla \mu^e) \varepsilon + \phi_j^e(\nabla \mu^e) \varepsilon \]

Inserting equation (1.11) in the equation (3.55), we get

\[ \psi_j(x+\mu\varepsilon)B_{s}\mu^a = \psi_j B^a \mu^b + \mu^k \{-\Sigma_{h} \eta^a_i \delta^k_{h} + \sum_{b} \eta^a_b \nabla \xi^a_{i} \}
\]

- \( (g^b_k - \sum_{a} \eta^a_k \eta^a_b ) \Sigma_{b} \xi^a_{b} B_{s}^a \mu^e + \mu^k (\nabla \psi_j) (\nabla \mu^e) \varepsilon + \phi_j^e(\nabla \mu^e) \varepsilon \)

Substituting equations (1.3), (1.4), (1.5), (1.6), (1.7), (1.8) and (1.10) in equation (3.63), we get

\[ \psi_j(x+\mu\varepsilon)B_{s}\mu^a = \psi_j B^a \mu^b + \mu^k \{-\Sigma_{h} \eta^a_i \delta^k_{h} + \sum_{b} \eta^a_b \nabla \xi^a_{i} \}
\]

By virtue of equations (2.2), (2.4), (3.6) and (3.57), we get

\[ \psi_j(x+\mu\varepsilon)B_{s}\mu^a = \psi_j B^a \mu^b - \sum_{b} \eta^a_b \Sigma_{b} \xi^a_{b} \nabla \mu^e + \phi_j^e(\nabla \mu^e) \varepsilon \]

On making use of equations (2.22), (2.23), (2.23) and (3.58), we have

\[ \psi_j(x+\mu\varepsilon)B_{s}\mu^a = \psi_j B^a \mu^b - \sum_{b} \eta^a_b \Sigma_{b} \xi^a_{b} \nabla \mu^e + \phi_j^e(\nabla \mu^e) \varepsilon \]

Substituting equations (1.3), (1.4), (1.5), (1.6), (1.7), (1.8) and (2.11) in equation (3.63), we obtain

\[ \psi_j(x+\mu\varepsilon)B_{s}\mu^a = \psi_j B^a \mu^b + \mu^k \{-\Sigma_{h} \eta^a_i \delta^k_{h} + \sum_{b} \eta^a_b \nabla \xi^a_{i} \}
\]

By virtue of equations (2.22), (2.23), (3.8) and (3.57), we get

\[ \psi_j(x+\mu\varepsilon)B_{s}\mu^a = \psi_j B^a \mu^b + \mu^k \{-\Sigma_{h} \eta^a_i \delta^k_{h} + \sum_{b} \eta^a_b \nabla \xi^a_{i} \}
\]

With the help of equation (3.28), we have

\[ \psi_j^h(x+\mu\varepsilon)C^{h}_{y} = \{\psi_j^h + (\mu^k \phi_j^h) \varepsilon\} + \{C^{h}_{y} - (\Gamma^{h}_{j} \mu^j C^{h}_{y} \varepsilon + (B^{h}_{a} \tau^{a}_{y} + C^{h}_{a} \tau^{a}_{y}) \varepsilon\}
\]

Inserting the equations (1.3), (1.4), (1.5), (1.6), (1.7), (1.8) and (1.10) in the equation (3.62), we obtain

\[ \psi_j^h(x+\mu\varepsilon)C^{h}_{y} = \{\psi_j^h + (\mu^k \phi_j^h) \varepsilon\} + \{C^{h}_{y} - (\Gamma^{h}_{j} \mu^j C^{h}_{y} \varepsilon + (B^{h}_{a} \tau^{a}_{y} + C^{h}_{a} \tau^{a}_{y}) \varepsilon\}
\]

Inserting equations (2.2), (2.4), (2.28), (2.29) and (2.33) in equation (3.63) and using contraction, we get

\[ \psi_j^h(x+\mu\varepsilon)C^{h}_{y} = \{\psi_j^h + (\mu^k \phi_j^h) \varepsilon\} + \{C^{h}_{y} - (\Gamma^{h}_{j} \mu^j C^{h}_{y} \varepsilon + (B^{h}_{a} \tau^{a}_{y} + C^{h}_{a} \tau^{a}_{y}) \varepsilon\}
\]

In this regard, we take

\[ \mu^k \phi_j^h \varepsilon = (\mu^a B^a \phi_j^h + \mu^c C^{y}_{c})
\]

By virtue of equations (2.22), (2.23) and (3.65), we get

\[ \mu^k \phi_j^h \varepsilon = (\mu^a B^a \phi_j^h + \mu^c C^{y}_{c})
\]

From equations (3.64) and (3.66), we get

\[ \psi_j^h(x+\mu\varepsilon)C^{h}_{y} = \{\psi_j^h + (\mu^k \phi_j^h) \varepsilon\} + \{C^{h}_{y} - (\Gamma^{h}_{j} \mu^j C^{h}_{y} \varepsilon + (B^{h}_{a} \tau^{a}_{y} + C^{h}_{a} \tau^{a}_{y}) \varepsilon\}
\]
Using relations $\eta^\alpha_z F^z_x = 0$ and $\xi^y_x F^x_y = 0$ in equation (3.67), we obtain

\begin{align*}
\delta\eta^i_j(x + \mu \epsilon)C^i_j y &= F^z_y \{C^i_z - (\Sigma \beta \eta^a_x) F^b_a y \mu^a \epsilon \} (1 + \tau^z_y \epsilon) + \{ F^b_a y \} \\
&- (2 \xi^z_y \eta^a_x y \mu^a \epsilon \} \tau^z_y \epsilon) - C^b_y \Gamma^i_y \mu^a \epsilon + ^y \mu \epsilon ^i F^a_b C^i_a \epsilon.
\end{align*}

\section*{4. INFINITESIMAL VARIATIONS OF TANGENTIAL SUBMANIFOLDS}

\textbf{Definition 4.1:}

The submanifold $M^n$ of a P-Sasakian manifold $M^{2m+1}$ is said to be tangent or tangential to the structure vector field $\mu^i$ if the following conditions are satisfied.

\begin{align*}
(4.1) \quad f^a = 0 \quad &\text{and} \quad f_x = 0 \\
(4.3) \quad \nabla_c f^a = 0 \quad &\text{and} \quad \nabla^c f_x = 0 \\
(4.5) \quad \delta f^a = 0 \quad &\text{and} \quad \delta f_x = 0
\end{align*}

It is also easy to verify that

\begin{align*}
(4.7) \quad f^a_c = 0 \\
(4.9) \quad \nabla_c f^a = f^a_c
\end{align*}

By virtue of equations (2.16) and (4.1), we obtain

\begin{align*}
(4.8) \quad f^a_c f^x = 0 \\
(4.10) \quad f^a = - h^a_{ab} f^b
\end{align*}

\textbf{Definition 4.2:}

The submanifold $M^n$ of a P-Sasakian manifold $M^{2m+1}$ is said to be tangent or tangential to the structure vector field $\mu^i$ if the following conditions are satisfied.

\begin{align*}
(4.11) \quad \mu^x = 0 \quad &\text{and} \quad \mu_x = 0 \\
\text{Wherein} \quad \mu_x = g_{xy} \mu^y
\end{align*}

It is also easy to verify that

\begin{align*}
(4.13) \quad \delta \mu^x = 0 \quad &\text{and} \quad \delta \mu_x = 0 \\
(4.15) \quad \nabla_c \mu^x = 0 \quad &\text{and} \quad \nabla^c \mu_x = 0
\end{align*}

In this regard, we have the following theorems:

\textbf{Theorem 4.1:}

If the submanifold $M^n$ tangent to the structure vector field $\mu^i$ of a P-Sasakian manifold $M^{2m+1}$, then the equation (3.33) reduces in the following form

\begin{align*}
(4.17) \quad \tau^x = - h^x_{ab} \mu^b
\end{align*}

\textbf{Proof:}

It is obvious from equations (3.33), (4.12) and (4.16).
Theorem 4.2:
If the submanifold of a P-Sasakian manifold is to be tangential to structure vector field $\mu^i$, then it is necessary and sufficient that the relation
\begin{equation}
(4.18) \quad \tau_a = h^{i}_{ab} \mu^b
\end{equation}
holds.

Proof:
It is obvious from equations (3.39), (4.11) and (4.15).

Suppose that the submanifold $M^n$ is tangent to the structure vector field $f^i$ and $\mu^i$, then

Using equation (4.11), the equation (3.43) reduces in the following form
\begin{equation}
(4.19) \quad \delta f^b_a = \{(f^b_c \mu^c - \mu^b f^c_a) + (\nabla^b \mu^c) f^a_c - (\nabla^c \mu^a) f^b_c - h^b_{bc} f^c_a \mu^b + h^b_{bc} f^c_b \mu^c \} \varepsilon
\end{equation}

From equations (4.10) and (4.19), we have
\begin{equation}
(4.20) \quad \delta f^b_a = \{(f^b_c \mu^c - \mu^b f^c_a) + (\nabla^b \mu^c) f^a_c - (\nabla^c \mu^a) f^b_c - h^b_{bc} f^c_a \mu^b + h^b_{bc} f^c_b \mu^c \} \varepsilon
\end{equation}

As a consequence of equations (4.17), (4.18) and (4.19) yields
\begin{equation}
(4.21) \quad \delta f^b_a = \{(f^b_c \mu^c - \mu^b f^c_a) + (\nabla^b \mu^c) f^a_c - (\nabla^c \mu^a) f^b_c - h^b_{bc} f^c_a \mu^b + h^b_{bc} f^c_b \mu^c \} \varepsilon
\end{equation}

Using equations (4.1) and (4.11), the equation (3.44) reduces in the form
\begin{equation}
(4.22) \quad \delta f^b_a = \{(f^b_c \mu^c) f^c_a + h^b_{bc} f^c_a \mu^b - h^b_{bc} f^c_b \mu^c \} \varepsilon
\end{equation}

By virtue of equations (4.10) and (4.23), we obtain
\begin{equation}
(4.23) \quad \delta f^b_a = \{(f^b_c \mu^c) f^c_a + h^b_{bc} f^c_a \mu^b - h^b_{bc} f^c_b \mu^c \} \varepsilon
\end{equation}

Inserting equation (4.18) into the equation (4.23), we get
\begin{equation}
(4.24) \quad \delta f^b_a = \{(f^b_c \mu^c) f^c_a + h^b_{bc} f^c_a \mu^b - h^b_{bc} f^c_b \mu^c \} \varepsilon
\end{equation}

From equations (4.10) and (4.25), we get
\begin{equation}
(4.25) \quad \delta f^b_a = \{(f^b_c \mu^c) f^c_a + h^b_{bc} f^c_a \mu^b - h^b_{bc} f^c_b \mu^c \} \varepsilon
\end{equation}

Using equations (4.1) and (4.11), the equation (3.48) becomes
\begin{equation}
(4.26) \quad \delta f^b_a = \{- (\nabla^b \mu^c) f^c_a + h^b_{bc} f^c_a \mu^b - h^b_{bc} f^c_b \mu^c \} \varepsilon
\end{equation}

By virtue of equations (4.17) and (4.27), we get
\begin{equation}
(4.27) \quad \delta f^b_a = \{- (\nabla^b \mu^c) f^c_a + h^b_{bc} f^c_a \mu^b - h^b_{bc} f^c_b \mu^c \} \varepsilon
\end{equation}

In view of equations (4.1) and (4.11), the equation (3.49) assumes the form
\begin{equation}
(4.28) \quad \delta f^b_a = \{- (\nabla^b \mu^c) f^c_a - f^a_y x^b f^c_x + f^b_x \tau^a_{xy} \} \varepsilon
\end{equation}

From equations (4.10) and (4.29), we get
\begin{equation}
(4.29) \quad \delta f^b_a = \{- (\nabla^b \mu^c) f^c_a - f^a_y x^b f^c_x + f^b_x \tau^a_{xy} \} \varepsilon
\end{equation}

By virtue of equations (4.17), (4.18) and (4.29), we obtain
\begin{equation}
(4.30) \quad \delta f^b_a = \{- (\nabla^b \mu^c) f^c_a - f^a_y x^b f^c_x + f^b_x \tau^a_{xy} \} \varepsilon
\end{equation}

In view of equations (4.10) and (4.31), we get
\begin{equation}
(4.31) \quad \delta f^b_a = \{- (\nabla^b \mu^c) f^c_a - f^a_y x^b f^c_x + f^b_x \tau^a_{xy} \} \varepsilon
\end{equation}

By virtue of equations (4.17), (4.18) and (4.29), we obtain
\begin{equation}
(4.32) \quad \delta f^b_a = \{- (\nabla^b \mu^c) f^c_a - f^a_y x^b f^c_x + f^b_x \tau^a_{xy} \} \varepsilon
\end{equation}
Using equations (4.1) and (4.11), the equation (3.53) reduces in the following form

\[ \delta f^a = \{f^a_{b\mu} - (\nabla_{b\mu} f)^b\} \epsilon \]

In view of equations (4.9) and (4.33), we obtain

\[ \delta f^a = \{f_{b\mu}^a - (\nabla_{b\mu} f)^b\} \epsilon \]

**Theorem 4.3:**

For the infinitesimal variations of tangential submanifolds of a P-Sasakian manifold, it is necessary and sufficient that \( \tau^x_{b} f^a - h^x_{b\mu} f^d = 0 \).

**Proof:**

Using the equations (4.1) and (4.11), the equation (3.54) becomes reduces into the following form

\[ \delta f^a = -\{f^a_{b\mu} + \tau^x_{b} f^d\} \epsilon \]

In view of equations (4.10) and (4.35), we get

\[ \delta f^a = \{f^x_{b\mu} f^b - \tau^x_{b} f^d\} \epsilon \]

By virtue of equations (4.5) and (4.36), we obtain

\[ \tau^x_{b} f^d - h^x_{b\mu} f^a = 0 \]

From equations (4.18) and (4.35), we get

\[ \delta f^a = -\{f^a_{b\mu} + h^x_{b\mu} f^d\} \epsilon \]

On making use of equation (4.10) in equation (4.38), we obtain

\[ \delta f^a = \{f^x_{b\mu} f^b - h^x_{b\mu} f^d\} \epsilon \]

In view of equations (4.5) and (3.54), we get

\[ f^x_{b\mu} - f^x_{b\mu} f^b - \tau^x_{b} f^d - \tau^x_{y} f^d = 0 \]

From equations (4.5) and (4.35), we follows

\[ f^x_{b\mu} + \tau^x_{b} f^d = 0 \]

As a consequence of equations (4.5) and (4.38), we have

\[ f^x_{b\mu} = -h^x_{b\mu} f^d \]

From equations (4.5) and (4.39), we obtain

\[ h^x_{b\mu} f^c - h^x_{b\mu} f^d = 0 \]

Consequently, from equations (4.11), (4.15) and (3.60) follows that

\[ \phi_j(x + \mu \epsilon) B_{\xi a}^i = F_{a B}^i B_{\xi b}^i + \{(\nabla_c B_{\xi b}^i)_{\mu} F^c_{a B}^i + (\nabla_{a \mu} c) F_{a B}^i B_{\xi b}^i\} \epsilon \]

In view of equations (4.11) and (4.17), the equation (3.68) assumes the form

\[ \phi_j(x + \mu \epsilon) C_{\xi y}^i = F_{a B}^i C_{\xi \mu a}^i \{1 + \tau^x_{y} \epsilon \}
-\{F_{a B}^i - (\nabla_{a \mu} c) C_{\xi \mu a}^i \} F_{a B}^i \epsilon
- C_{\xi \Gamma_{j}^k} F_{a B}^i B_{\xi b}^i \epsilon. \]
REFERENCES


[6]. S.Sasaki : Almost contact manifolds, Lecture Notes,1,Tohoku University, (1965).


Received: September, 2010