Measuring Risk Element Criticality in a Fuzzy Project Network Using Trapezoidal Fuzzy Number Method

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Abstract
Risk management is a process aimed at enhancement, developing the level of security in an organization. It gives the organization a broad view of the risks that can affect its productivity and performance, and enables it to make appropriate risk management decisions. This paper presents a new fuzzy method for measuring risk element criticality in a fuzzy project network. There have been several attempts in the literature to apply fuzzy numbers to the critical path method. We have modified the former presented approaches to this problem using new trapezoidal fuzzy number method. At the end of the paper, an example is presented. The proposed method in this paper is more effective in determining the activity criticality, finding the fuzzy critical path, and obtaining schedule risk.

1. Introduction
Critical path method (CPM) in Program evaluation and review technique (PERT) has been demonstrated to be a useful tool in the project time control [1-3], this technology shows that the keystone of the project control is the critical path because the activities in CPM could induce the project success or failure, and the total duration time in CPM is considered as the best times of the project. When the activity times in the project are deterministic and known, CPM has been demonstrated to be an efficient manner. However, there are many cases where the activity times may not be presented in a precise manner since the risk factor exists. To deal quantitatively with imprecise data, probability theory [4] and
Monte Carlo Simulation [5,6] are employed. But in these analysis, the probability distributions of each activities is needed, it is difficult to use in some situations when the priori data of the activity probability distributions are absence. Therefore, there are some detailed critiques of PERT can be found in some work [7,8].

An alternative way to deal with imprecise data is to employ the fuzzy numbers. The main advantages of this methodology are that it does not require prior predictable regularities or posterior frequency distributions, and it can deal with imprecise input information based on the human subjective judgment. Stefan [8] gives the related definitions and theorems for the fuzzy critical path problem, but the method to calculated the critical path is not present. Liu [9] only gives an approximation method for fuzzy critical path, and Liu [10] proposes a deadline based method, although the most critical path is given, it need give the deadline. However, the shortest total time is the key of the project in PERT, and with the uncertain of the fuzzy number, the critical path may change with different conditions, it is most important for decision makers to how to give these information. The interval number critical path problem is studied by Stefan [11] and Liu [12,13], and the interval number is a special situation of the fuzzy number, but it is not solved the fuzzy number critical path problems absolutely. Therefore, a new trapezoidal fuzzy number method for measuring risk element criticality is proposed.

2. Preliminaries

In this section some basic definitions, arithmetic operations, ranking functions, risk analysis are reviewed.

2.1 Fuzzy concepts

**Definition 1 : Fuzzy Set**

Let the universe of discourse be \( X = \{x\} \). A fuzzy set \( A \) on \( X \) is a mapping which assigns to each element \( x \in X \) a real number in the interval \([0,1]\) expressing the membership of \( x \) in \( A \).

**Definition 2 : Membership Function**

The membership function, \( \mu_A(x) \), represents the grade of membership of \( x \) in \( A \), denoted as \( \mu : X \rightarrow [0,1] \), or \( \mu_A(x) \), for all \( x \in X \).

**Definition 3 : \( \alpha \)-level set**

The \( \alpha \)-level set is a crisp set \( A_\alpha \) such that
Measuring risk element criticality

\[ A_\alpha = \{ x \in X \mid \mu_{\tilde{A}}(x) \geq \alpha \} \text{ for } \alpha \in [0,1] \]

**Definition 4 : The Support of \( \tilde{A} \)**

\[ S(\tilde{A}) = \{ x \in X \mid \mu_{\tilde{A}}(x) > 0 \} \]

**Definition 5 : Normalization**

A normalized fuzzy set satisfies \( \mu_{\tilde{A}}(x) = 1 \), there exists \( x \in X \).

**Definition 6 : Modal value**

The single value \( m \), such that \( \mu_{\tilde{A}}(x) = 1 \), is called the modal value of the fuzzy set.

**Definition 7 : Fuzzy number**

A fuzzy number \( \tilde{M} \) is a normalized fuzzy set whose membership function is unimodal and upper-semicontinuous.

**Fuzzy Arithmetic :**

If \( \tilde{M} \) and \( \tilde{N} \) are trapezoidal fuzzy numbers,

\[ \tilde{M} = (u_1, u_2, u_3, u_4); \tilde{N} = (v_1, v_2, v_3, v_4), \]

then

\[ \tilde{M} \oplus \tilde{N} = (u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4) \]

and

\[ \tilde{M} \Theta \tilde{N} = (u_1 - v_4, u_2 - v_3, u_3 - v_2, u_4 - v_1). \]

**2.2 Comparison of Trapezoidal fuzzy numbers [14]**

Chen and Chen [14] presented a method for ranking fuzzy numbers. It can evaluate the ranking order between fuzzy numbers based on centre of gravity(COG) points and standard deviations of fuzzy numbers. The ranking value

\[ \text{Rank}(\tilde{A}) = x^*_A + \left(1 - y^*_A\right)^\delta, \]

(1)
where the values \( x^*_A \) and \( y^*_A \) are the COG point \( (x^*_A, y^*_A) \) of the generalized fuzzy number \( A \) calculated as follows:

\[
y^*_A = \begin{cases} 
1 \times \frac{a_i - a_2}{a_4 - a_i} + 2 & \text{if } a_i \neq a_4 \\
6 & \text{if } a_i = a_4 
\end{cases}
\]

(2)

\[
x^*_A = \frac{y^*_A(a_4 + a_2) + (a_4 + a_i)(1 - y^*_A)}{2}
\]

(3)

The value of \( S^*_A \) of Eq. (1) denotes the standard deviation of the generalized fuzzy number \( A \), and is defined as follows:

\[
S^*_A = \sqrt{\frac{\sum_{i=1}^{4} (a_i - \bar{a})^2}{4 - 1}} = \sqrt{\frac{\sum_{i=1}^{4} (a_i - \bar{a})^2}{3}}
\]

(4)

where \( \bar{a} \) denotes the mean value of the values \( a_1, a_2, a_3 \) and \( a_4 \), that is, \( \bar{a} = \frac{a_1 + a_2 + a_3 + a_4}{4} \). The larger the value of Rank \( (A) \), the better the ranking of \( A \).

2.3 Risk, Risk Analysis and Risk Management

Definition 8: Risk
A risk is an event, which is uncertain and it has a negative impact on some endeavor.

Example: To a life insurance company the timing of deaths of its policyholders are risks. The company never knows precisely who among their insurers is going to die in a given period of time (uncertainty) and each death costs them a payout equal to the face value of the policy (negative impact on profitability).

Risk Analysis and Risk management:
Risk analysis is the process of assessing risks, while risk management uses risk analysis to devise management strategies to reduce or ameliorate risk. In project
management, these techniques are used to address the questions “how long will this project eventually take?” (schedule risk), “how much will it finally cost?” (cost risk), and “will its product perform according to specifications?” (performance risk).

Risk analysis is the process of quantitatively or qualitatively assessing tasks. This involves an estimation of both the uncertainty of the risk and of its impact. Again, an insurance company can estimate the number of deaths in a given period based on demographic information about their insurers; this estimate, coupled with information about their policies, in turn allows them to estimate the amount of money they will have to pay off in the time period in question. In general, these estimates will not match the exact amount of money paid out, but a key part of the uncertainty analysis will allow the insurance company to have an idea of how likely different payoffs are in a range around their estimate.

Risk management is the practice of using risk analysis to devise management strategies to reduce or ameliorate risk. In order to deal with an estimated payoff, the insurance company may revise its investment strategy, change eligibility for insurance, target different populations for sales of policies, or even cancel policies if possible to control the amount of money they expect to pay out and ensure that they make a profit.

**Example**: In engineering design, reliability estimates of different parts are combined with an assessment of the impact on system performance of the failure of the parts. This analysis has in turn been used to direct resources for modification and redesign to those areas of aircraft, nuclear reactors, and other complex man-made systems where improvements have the most effect on reducing potential failures. Success in this area has led to expanding the practice of assessing and managing risks to economies and eco-systems. In this report we will be concerned with the use of these techniques in managing complex projects, where some of the important questions are “how long will this project eventually take?”, “how much will it finally cost?”, and “will its product perform according to specifications?”

### 2.4 Fuzzy project characteristics

In accordance with Critical Path Method (CPM), the forward pass yields the fuzzy earliest start and earliest finish times:

\[
\tilde{S}_i = \max_{j \in P(i)} \left\{ \tilde{S}_j^e \oplus \tilde{d}_j \right\}
\]

\[
\tilde{F}_i = \tilde{S}_i^e \oplus \tilde{d}_i
\]
where $S_i^e$ is the fuzzy earliest start time (with $S_i^e = (0,0,0,0)$ at the initial node $i = 1$), $F_i^e$ is the fuzzy earliest finish time (with $F_i^e$ equal to the project network completion time $T_{end}$ at the ending node $i = E$), $P(i)$ is the set of predecessors for activity $i$, and $d_i$ is the operation time for activity $i$.

The backward pass is performed to calculate the fuzzy latest start and latest finish times:

$$
F_i^l = \min_{j \in S(i)} \left( F_j^l \Theta d_j \right) \quad (7)$$

$$
S_i^l = F_i^l \Theta d_i \quad (8)
$$

where $F_i^l$ is the fuzzy latest finish time (with $F_i^l = T_{end}$ when $i = E$), $S_i^l$ is the fuzzy latest start time, and $S(i)$ is the set of successors of activity $i$.

The Fuzzy CPM implementation employs the ranking method [14] to compare fuzzy numbers and compute $S_i^e$, $S_i^l$, $F_i^e$ and $F_i^l$ for each activity $i$.

In traditional PERT, the float time for each activity is either the difference between the latest and earliest starting times or the difference between the latest and earliest finishing times. Once $S_i^e$, $S_i^l$, $F_i^e$ and $F_i^l$ have been determined for the $i^{th}$ activity, the fuzzy float time is either

$$
m_i = S_i^l \Theta S_i^e \quad (9)$$

Or

$$
m_i = F_i^l \Theta F_i^e \quad (10)
$$

3. Proposed Fuzzy critical path method with project characteristics

Consider the fuzzy project network with $n$ vertices, where the duration time $(t_{ij})$ of each activity $(i,j)$ in a fuzzy project network is represented by a trapezoidal fuzzy number. Fig. 1 represents example of a fuzzy project network with 5 vertices. Table I represents the activities of fuzzy project network with fuzzy project duration.

**Step 1:** First we have to set a table of order $(n-1) \times (n-1)$. The row numbers are i-index and the column numbers are j-index where $i=1,2,\ldots n-1$; $j=2,3,\ldots n$. 
Step 2: Make the entries $T_{ij}$, the maximum fuzzy trapezoidal time required from project start to the finish of the activity $i \rightarrow j$. Enter the first row entries in the table i.e. the times $t_{1j}$. Then move to fill in the entries of the second row ($i = 2$) by adding the preceding time of the second node and $t_{2j}$. Then move filling the entries of third row ($i = 3$) by adding the preceding time of the third node and $t_{3j}$ i.e. $T_{3j} = t_{13} + t_{3j}$ and so on. If we get more than one time value (paths) then we can take the maximum. These values are presented in Table-II for our example.

Step 3: To identify the critical path by backing up from the known ending point. Start by selecting the entry with the largest fuzzy value from the last column ($j = n$). This value is the project’s fuzzy completion time. The row number in which the largest value in the last column is located gives the index of the tail node of the last activity; therefore, the head node of the next-to-last activity on the critical path is known. If the column-$n$ largest value in the row $k$, then column $k$ is examined and the largest value in that column is sought. This process is repeated until the start node is reached, and in so doing, a list of all activities along the critical path is generated.

In our example, the fuzzy critical path is 1-2-3-5. The trapezoidal fuzzy numbers in the Table II are darken.

Step 3: Form another $(n-1) \times (n-1)$ order table for computing free slack times. Select the entry with the largest value $T_{end} = T_{kj}$ where $k = 2, \ldots, n-1$; $j = n$, from the last column of the Table-II which gives the projects completion time. We have to calculate the free slack times by using the formula given below:

$$FS_{ij} = \max\{T_{kj};\text{all } k\} - T_{ij},$$

where $T_{ij} =$ all the row $i$ entries

$= \text{the fuzzy activity time } t_{ij} + \text{the maximum fuzzy completion time in column } i$

These values are presented in Table-III.

Step 4: Form another $(n-1) \times (n-1)$ order table for computing total slack times by using

$$TS_{ij} = FS_{ij} + \min\{TS_{kj};\text{all } k>j\}.$$ 

The bottom row is unique i.e. $TS_{ij} = FS_{ij}$ and is used to start the calculation and we have to put those entries in Table-IV.

Step-5: Finding the total latest start and finishing times:

Form another table of order $(n-1) \times (n-1)$ whose entries would be $L_j-t_{ij}$ (the latest starting times) and the minimum element of row $i$ would give the value of $L_i$(the latest finish times). Thus $L_j$ - $t_{ij}$ = all the column $j$ entries = minimum element of row $j - t_{ij}$

This process is continue along the decreasing order of column number until the first column $j = 2$ is reached. These values are presented in Table-V.
Fig. 1 Fuzzy project network

Table I: Activity duration of each activity in a fuzzy project network

<table>
<thead>
<tr>
<th>Activity</th>
<th>Activity duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-2</td>
<td>(10,15,15,20)</td>
</tr>
<tr>
<td>1-3</td>
<td>(30,40,40,50)</td>
</tr>
<tr>
<td>2-3</td>
<td>(30,40,40,50)</td>
</tr>
<tr>
<td>1-4</td>
<td>(15,20,25,30)</td>
</tr>
<tr>
<td>2-5</td>
<td>(60,100,150,180)</td>
</tr>
<tr>
<td>3-5</td>
<td>(60,100,150,180)</td>
</tr>
<tr>
<td>4-5</td>
<td>(60,100,150,180)</td>
</tr>
</tbody>
</table>

Table II: Calculation of times (Tij)

<table>
<thead>
<tr>
<th>Nodes</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(10,15,15,20)</td>
<td>(30,40,40,50)</td>
<td>(15,20,25,30)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>(40,55,55,70)</td>
<td></td>
<td>(70,115,165,200)</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td>(100,155,205,250)</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td>(75,120,175,210)</td>
</tr>
</tbody>
</table>

Fuzzy critical path is 1-2-3-5
Table III : Free slack times

<table>
<thead>
<tr>
<th>Nodes</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-10,0,0,10)</td>
<td>(-10,15,15,40)</td>
<td>(-15,-5,5,15)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>(-30,0,0,30)</td>
<td></td>
<td>(-100,-10,90,180)</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td>(-150,-50,50,150)</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td>(-110,-20,85,175)</td>
<td></td>
</tr>
</tbody>
</table>

Table IV : Total Slack times

<table>
<thead>
<tr>
<th>Nodes</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-190,-50,50,190)</td>
<td>(-160,-35,65,190)</td>
<td>(-275,-75,140,240)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>(-180,-50,50,180)</td>
<td>(-250,-60,140,330)</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td>(-150,-50,50,150)</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td>(-260,-70,135,325)</td>
<td></td>
</tr>
</tbody>
</table>

Table V : Total latest start and finishing times

<table>
<thead>
<tr>
<th>Nodes</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-10,0,0,10)</td>
<td>(-40,-25,-25,-10)</td>
<td>(-20,-10,-5,5)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>(-10,15,15,40)</td>
<td>(-140,-95,-45,10)</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td>(-80,5,105,190)</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td>(-105,-30,75,150)</td>
<td></td>
</tr>
</tbody>
</table>

Conclusion

Several research have investigated the critical path analysis in the project network with fuzzy activity times. A new fuzzy method for measuring risk element criticality in a fuzzy project network has been proposed. A numerical example has particularly provided to explain the proposed procedure in detail.

References


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