A Non-Cooperative Game for Two Typologies Users
Routing in Networks with Side Constraints

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Abstract

We consider a network shared by non-cooperative two types of users, priority users and non-priority users. The first type choose links that can accommodate their performance requirements and reserve the necessary resources so as to minimize their costs where their strategy sets are coupled but independent of those of non-priority users, and the spare resources (if it exist) used by the non-priority users. We study non-cooperative routing, we address the basic questions of existence and uniqueness of equilibrium. We show that equilibria indeed exist but it may not be unique due to the multicriteria nature of the problem. We are able, however, to obtain uniqueness in some weaker sense under appropriate conditions, we show that the link utilizations are uniquely determined at equilibrium and the normalized Nash equilibrium is unique.

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1 INTRODUCTION

The complexity of high-speed, large-scale networks calls for decentralized control algorithms, where control decisions are made by each user independently, according to its own individual performance objectives. Such networks are henceforth called noncooperative, and game theory [6], [4] provides the systematic framework to study and understand their behavior. The operating points of a noncooperative network are the Nash equilibria of the underlying game, that is, the points where unilateral deviation does not help any user to improve its performance.

QoS routing mechanisms allow users to identify paths that can accommodate their performance requirements (such as minimum delays) and reserve the necessary resources. In real-time applications, however, an application may have several criteria for quality of service. It might be sensitive to delays and losses, or it might seek to minimize some cost imposed on the use of network resources. In the presence of several users each with several objectives, who determine the routes for flows they control, this gives rise to a noncooperative multicriteria game. As is often the case in today’s networks, quality of service of an application is often given through an upper bound on some performance measure (delay, loss rate or jitter, see e.g. [9]). In the present work, we consider a communication network shared by two typologies of selfish users: priority users and non-priority users. For two typologies of users, each user seeks to optimize its own performance by controlling the routing of its given flow demand, assuming that the priority users choose the necessary capacity in the links, while respecting the constraints on the load at each link, and the non-priority users use the spare links capacity (if it exists). An appropriate framework for modeling this situation is that of non-cooperative game theory, and the constrained Nash equilibrium is the corresponding solution concept.

The authors in [3] have investigated the special topology of parallel links without flow control, they showed through a simple example that there may exist several equilibria, although in the absence of side constraints there would be a single equilibrium [7]. Then they showed the uniqueness of normalized equilibrium as in [8] for the constrained parallel links problem. In this paper, our objective is to study the existence and uniqueness of coupled Nash equilibria in the case where the network shared by two typologies of players: priority and non-priority users. We show by a simple example of parallel links that there may be several such equilibria. In absence of side constraints there would be a single equilibrium [7]. We establish existence results, we present a weak uniqueness result for the parallel link topology, and we show that although the equilibrium may not be unique, the link’s utilization at equilibrium is unique under some conditions. We establish the uniqueness of the normalized Nash equilibrium for the case of parallel links.
The paper is organized as follows. In section 2, the model is presented. In section 3 the existence of coupled Nash equilibrium and normalized equilibrium for parallel links are established. In section 4, we study the uniqueness of equilibria in parallel links topology. In section 5, we study the uniqueness of normalized Nash equilibrium. The paper ended with some conclusion in section 6.

2 MODEL AND PROBLEM FORMULATION

2.1 Model

The adopted model is a network of parallel links where a source node S and a destination node D are interconnected by a set of parallel communication links \( L = \{1, 2, ..., L\} \). Each link is characterized by a fixed overall capacity to be shared among the users.

We are given a set \( I^p = \{1, 2, ..., I\} \) of priority users. With each user \( i \), we associate the source S and the destination D, and a throughput demand \( r^i \).

A group \( I^{np} = \{1, 2, ..., K\} \) of non-priority users accesses the network through the common source node S. With each user \( k \), we associate a throughput demand \( r^k \).

The characteristics of the underlying game depend on the level of priority (QoS) adopted at the source node for each typology of players.

Let \( c_l \) denote the capacity of link \( l \in L \).

Two typologies of players participate to the non-cooperative game taking place at the source node S: the priority and the non-priority users.

The objective of every priority user \( i \in I^p \) is to ship from the source S to the destination D his own total demand \( r^i \) by splitting it among the L links.

Each priority user \( i \in I^p \) chooses links that can accommodate their performance requirements and reserves the necessary resources, and defines his own strategy by deciding the amount of flow he or she will send on every link \( l \in L \) where the strategy sets of priority users are coupled but independent of that of non-priority users. More formally, a strategy of player i is given by the vector \( f^i = (f^i_1, f^i_2, ..., f^i_L) \in R \). The quantity \( f^i_1 \), referred to as elementary flow, represents the amount of flow sent by the user \( i \in I^p \) over the link \( l \in L \).

Moreover, \( f_l = \sum_{i \in I^p} f^i_l \) denotes the total flow on the link \( l \), caused by the I users. The vector grouping all the priority user strategies is denoted by \( f = (f^1, f^2, ..., f^I) \in R^{I \times L} \).

In order to benefit from the spare links capacity in a completely non-cooperative framework, we assume that the objective of every non-priority user \( k \in I^{np} \) is to ship from the source S to destination D her amount of traffic by splitting it among the spare link capacity \( (\sum_{l \in L} c^np_l = \sum_{l \in L} c_l - \sum_{l \in L} c^p_l) \).
Each non-priority player \(k \in I^{np}\) defines her own strategy \(g^k = (g^k_1, g^k_2, ..., g^k_L) \in R^L\), where the quantity \(g^k_l\), represents the amount of flow sent by the user \(k \in I^{np}\) over the link \(l \in L\). The collection of non-priority users strategies is denoted by \(g = (g^1, g^2, ..., g^K) \in R^{K \times L}\).

The vector \((f, g) \in R^m\), with \(m = L(I + K)\), grouping both the priority user strategy profile and the non-priority strategy profile is referred to as network strategy profile where both the priority users \(I^p\) and the non-priority players \(I^{np}\) simultaneously operate.

Each priority user \(i \in I^p\) aims at maximizing her degree of satisfaction related to the QoS level provided by the network, i.e. at finding an appropriate strategy \(f^i = (f^i_1, f^i_2, ..., f^i_L)\) minimizing his cost function \(J^{(i, p)}(f)\) which depends only on the strategies of the other priority users \(f^{-i} = (f^1, f^2, ..., f^{i-1}, f^{i+1}, ..., f^I)\).

Analogously each non-priority player \(k \in I^{np}\) aims at minimizing a cost function \(J^{(k, np)}(f, g)\) which depends on both the strategies of the other non-priority players \(g^{-k} = (g^1, g^2, ..., g^{k-1}, g^{k+1}, ..., g^K)\) and the strategy profile \(f = (f^1, f^2, ..., f^I)\).

Each minimization problem composing the game has to be solved meeting some specific constraints. In particular, the strategy \(f^i\) minimizing the cost function \(J^{(i, p)}(f)\) has to respect the following constraints:

\[
\begin{align*}
\begin{cases}
    f^i_l \geq 0, \forall i \in I^p, \forall l \in L &: \text{non-negativity constraint}; \\
    f^i_l \leq c_l - \sum_{j \in I^p \setminus \{i\}} f^j_l, \forall l \in L &: \text{capacity constraint}; \\
    \sum_{l \in L} f^i_l = r^i, \forall i \in I^p &: \text{demand constraint}.
\end{cases}
\end{align*}
\]

The constraints to be respected by the strategy \(g^k\) in minimizing the cost function \(J^{(k, np)}(f, g)\) are:

\[
\begin{align*}
\begin{cases}
    g^k_l \geq 0, \forall k \in I^{np}, \forall l \in L &: \text{non-negativity constraint}; \\
    g^k_l \leq c_l - \sum_{i \in I^p} f^i_l - \sum_{h \in I^{np} \setminus \{k\}} g^h_l, \forall l \in L &: \text{capacity constraint}; \\
    \sum_{l \in L} g^k_l = r^k, \forall k \in I^{np} &: \text{demand constraint}.
\end{cases}
\end{align*}
\]

The above-mentioned relations show that in the considered model there exists a coupling among the constraint sets of the two kinds of players (priority users and non-priority users), because of the presence of the second inequality in (2) which couples the strategies. Therefore, the NEP solutions of the game have to be searched within the coupled constraint set \(F\) defined by the sets of relations (1) and (2), i.e.:

\[
F = \{(f, g) \in R^{I^p \times L + K \times L} : 0 \leq f^i_l \leq c_l - \sum_{j \in I^p \setminus \{i\}} f^j_l, \sum_{l \in L} f^i_l = r^i, \sum_{l \in L} g^k_l = r^k \}
\]

\[
0 \leq g^k_l \leq c_l - \sum_{i \in I^p} f^i_l - \sum_{h \in I^{np} \setminus \{k\}} g^h_l, \forall i \in I^p, \forall l \in L, \forall k \in I^{np} \}
\]

We denote the overall throughput demand for priority users by \(R = \sum_{i \in I^p} r^i\) and, the overall throughput demand for non-priority users by \(\hat{R} = \sum_{k \in I^{np}} r^k\).
The stability constraints entails that the following conditions hold:

\[ R < C \quad \text{and} \quad \dot{R} < C - R, \quad (4) \]

where \( C = \sum_{l \in \mathcal{L}} c_l \) is the overall capacity.

Each network strategy profile \((f, g)\) is said to be admissible if it belongs to the coupled constraint set \( F \). Let \( F_1 \) be the set of strategies for priority user. An admissible strategy \((f^*, g^*)\) is a NEP if the following conditions hold:

\[
\begin{align*}
J^{(i,p)*} &= J^{(i,p)}(f^*) = \min_{f^i} \{ J^{(i,p)}(f^i, f^{-i}) : (f^i, f^{-i}) \in F_1 \}; \forall i \in I^p \\
J^{(k,np)*} &= J^{(k,np)}(f^*, g^*) = \min_{g^k} \{ J^{(k,c)}(f^*, g^k, g^{-k}) : (f^*, g^k, g^{-k}) \in F \}; \forall k \in I^{np}
\end{align*}
\]

(5)

2.2 Properties of the Cost Functions

We will assume that the priority users cost functions have the following properties.

\( G_1 : J^{(i,p)} \) is given as the sum of link costs, i.e. \( J^{(i,p)}(f) = \sum_{l \in \mathcal{L}} J_l^{(i,p)}(f_l) \),

\( G_2 : J^{(i,p)}(f) : R^{I.L} \rightarrow R^+ \) is continuous in \( f \),

\( G_3 : J_l^{(i,p)}(f_l, f_i) \) is convex in \( f_l \),

\( G_4 : J_l^{(i,p)}(f_l, f_i) \) is continuously differentiable with respect to \( f_l, f_i \), and the derivative \( K_l^{(i,p)} = \frac{\partial J_l^{(i,p)}}{\partial f_l} \) is a function of the two arguments \( f_l \) and \( f_i \) i.e. \( K_l^{(i,p)}(f_l, f_i) \),

\( G_5 : \lim_{f_l \rightarrow c_l} J_l^{(i,p)}(f_l, f_i) = \infty \).

Functions that satisfy assumptions \( G_1 - G_5 \) shall be referred to as type-G functions.

We will use the following set of assumptions (for the priority users cost functions).

\( A_1 : J_l^{(i,p)} \) is a function of two arguments, i.e. \( J_l^{(i,p)}(f_l, f_i) \),

\( A_2 : J_l^{(i,p)}(f_l, f_i) \) is increasing in two arguments \( f_l \) and \( f_i \),

\( A_3 : K_l^{(i,p)}(f_l, f_i) \) is increasing in each of its arguments and (due to \( G_3 \)) strictly increasing in the first one.

We refer to functions that satisfy assumptions \( A_1 - A_3 \) as type-A functions.

Moreover, we assume that the non-priority user cost functions have the following properties.

\( G'_1 : J_l^{(k,np)} \) is given as the sum of link costs \( J_l^{(k,np)}(f, g) = \sum_{l \in \mathcal{L}} J_l^{(k,np)}(f_l, g_l) \),

\( G'_2 : J_l^{(k,np)}(f_l, g_l) : R^{I.L+K.L} \rightarrow R^+ \) is continuous in \((f, g)\),

\( G'_3 : J_l^{(k,np)}(f_l, g^k_l, g_l) \) is convex in \( g^k_l \),
\( G'_4 : J^{(k, np)}_l \) is continuously differentiable in \( g^k_l \). We set \( K^{(k, np)}_l = \frac{\partial J^{(k, np)}_l}{\partial g^k_l} \).

\( G'_5 : \lim_{g_l \to c_l} J^{(k, np)}_l (f_l, g^k_l, g_l) = \infty. \)

Functions that comply with the aforementioned assumptions shall be referred to as type-\( G' \) functions.

We will refer to functions that meet the conditions of these three assumptions as type-\( A' \) functions.

\( A'_1 : J^{(k, np)}_l (f_l, g_l) \) is a function of three arguments, i.e. \( J^{(k, np)}_l (f_l, g^k_l, g_l) \),

\( A'_2 : J^{(k, np)}_l (f_l, g^k_l, g_l) \) is increasing in three arguments \( f_l, g^k_l \) and \( g_l \),

\( A'_3 : K^{(k, np)}_l (f_l, g^k_l, g_l) \) is increasing in each of its arguments and (due to \( G'_3 \)) strictly increasing in the second one.

**Remark 2.1.** Cost functions used in real networks are either related to actual pricing, or they are related to some performance measure such as expected delay. In the first case, a frequently used cost is that of linear link costs, i.e. for each priority user \( i \), \( J^{(i,p)}(f_i) = \sum_{l \in L} f_i T_l(f_l) \) where \( T_l(f_l) = a_l f_l + b_l \) [5] and for each non-priority user \( k \), \( J^{(k, np)}(f, g) = \sum_{l \in L} g_k T_l(f_l + g_l) \), where \( T_l(f_l + g_l) = a_l(f_l + g_l) + b_l \). When the costs represent delays they typically have the same form but with \( T_l(f_l) = (c_l - f_l)^{-1} + d_l \) and \( T_l(f_l + g_l) = (c_l - f_l - g_l)^{-1} + d_l \) where the first term represents queuing delay, with \( c_l \) standing for the queue capacity, and \( d_l \) represents the propagation delay related to link \( l \). The queuing delay is that of an M/M/1 queue operating under the FIFO regime (First In First Out, see [7]) or of an M/G/1 queue operating under the processor sharing regime. Other, more complicated costs can be found in [1].

### 3 EXISTENCE OF EQUILIBRIA

#### 3.1 Characterization of equilibria

We first consider the priority user group. In order to determine the admissible strategy \( f^{*i} \) allowing the minimization of her own cost function \( J^{(i,p)}(f) \), the user \( i \in I^p \) has to solve a constrained optimization problem of the form:

\[
\begin{align*}
\min_{f^i} & \quad J^{(i,p)}(f^i, f^{-i*}) \\
H(f^i, f^{-i*}) & = \sum_{l \in L} f^i_l - r^i = 0 \\
G(f^i, f^{-i*}) & = f^i_l + \sum_{j \in I^p \setminus \{i\}} f^j_l - c_l \leq 0
\end{align*}
\] (6)

A feasible routing strategy \( f^{*i} \) of the user \( i \), is a solution of the minimization problem (6) if and only if it satisfies the KKT conditions (8).
Likewise, each non-priority player $k \in I_{np}$, in order to determine the admissible strategy $g^*_k$, has to solve a constrained optimization problem of the form:

$$
\begin{align*}
\min_{g^k} & J^{(k,np)}(f^*, g^k, g^{-k*}) \\
H'(g^k, g^{-k*}) &= \sum_{l \in L} g^k_l - \hat{r}^k = 0 \\
G'(g^k, g^{-k*}) &= g^k_l + \sum_{h \in I_{np} \setminus \{k\}} g^h_l + \sum_{i \in I_p} f^i_l - c_l \leq 0
\end{align*}
$$

(7)

A feasible routing strategy $g^*_k$ of the user $k$, is a solution of the minimization problem (7) if and only if it satisfies the KKT conditions (9).

Extending these results to the $I + K$ players (I priority users and K non-priority users), the KKT conditions (8) and (9) are the necessary and sufficient conditions for an admissible network strategy profile $(f, g)$ to be a CNE of the overall non-cooperative game:

For every $i \in I_p$, there exists a set of Lagrange multipliers $\lambda^{(i,p)}$ and $\beta^{(i,p)}_l$ such that, for every link $l \in L$:

$$
\begin{align*}
K_l^{(i,p)}(f^i_l, f_l) - \lambda^{(i,p)} + \beta^{(i,p)}_l &= 0 \text{ if } f^i_l > 0 \\
K_l^{(i,p)}(f^i_l, f_l) - \lambda^{(i,p)} + \beta^{(i,p)}_l &\geq 0 \text{ if } f^i_l = 0 \\
\beta^{(i,p)}_l G(f^i, f^{-i}) &= 0 \\
G(f^i, f^{-i}) &\leq 0, f^i_l \geq 0 \\
\beta^{(i,p)}_l &\geq 0
\end{align*}
$$

(8)

And for every $k \in I_{np}$, there exists a set of Lagrange multipliers $\lambda^{(k,np)}$ and $\beta^{(k,np)}_l$ such that, for every link $l \in L$:

$$
\begin{align*}
K_l^{(k,np)}(f_l, g^k_l, g_l) - \lambda^{(k,np)} + \beta^{(k,np)}_l &= 0 \text{ if } g^k_l > 0 \\
K_l^{(k,np)}(f_l, g^k_l, g_l) - \lambda^{(k,np)} + \beta^{(k,np)}_l &\geq 0 \text{ if } g^k_l = 0 \\
\beta^{(k,np)}_l G(g^k, g^{-k*}) &= 0 \\
G'(g^k, g^{-k*}) &\leq 0, g^k_l \geq 0 \\
\beta^{(k,np)}_l &\geq 0
\end{align*}
$$

(9)

3.2 Normalized Nash Equilibrium

We now define a subclass of CNE, called normalized nash equilibrium.

**Definition 3.1.** A coupled Nash equilibrium $(f, g)$ is a normalized Nash equilibrium [8] if there exist a vector $\vec{\alpha} > 0$ where $\vec{\alpha} = (\alpha^1, \alpha^2, ..., \alpha^{I+K})$ and $\mathbf{0}$ is a vector of zeros, and some constants $\beta_l \geq 0, l \in L$, such that conditions (8)-(9) are satisfied with:

$$
\beta^j_l = \beta_l / \alpha^j, l \in L, j \in I_p \cup I_{np}
$$

Note that if a user’s weight $\alpha^j$ is greater than those of her competitors, then her corresponding Lagrange multipliers are smaller.
3.3 Existence of Equilibria

The proof of the existence of an equilibrium in the proposed game presents some difficulties related to the fact that the priority and non-priority cost functions may assume infinite values for network strategy profiles in the coupled constraint set $F$ (see properties $G_5$ and $G_5'$).

Properties $G_5$ and $G_5'$, along with the overall stability conditions (4), guarantee that at the equilibrium (if it exists) no saturation takes place on any link, i.e. that $f_l + g_l < c_l, \forall l \in \mathcal{L}$. Even though this assures that the costs of all players are finite at the equilibrium point, unfortunately, such a property can not be straightforwardly extended to all the points in $F$. Possible infinite costs in points of $F$ entail that the existence of the CNE can not be proved as a direct consequence of the game convexity [2] and [8].

Note that the game execution entails that each priority user $i \in I_p$ minimizes his cost function:

$$J^{(i,p)}(f) = \sum_{l \in \mathcal{L}} J^{(i,p)}_l(f_l),$$

and each non-priority player $k \in I_{np}$ minimizes his cost function:

$$J^{(k,np)}(f, g) = \sum_{l \in \mathcal{L}} J^{(k,np)}_l(f_l, g_l).$$

The minimization of the priority cost function $J^{(i,p)}(f)$ is equivalent to the maximization of the priority utility function $\hat{J}^{(i,p)}(f) = 1 / J^{(i,p)}(f)$.

Analogously, the minimization of the non-priority cost function $J^{(k,np)}(f, g)$ is equivalent to the maximization of the capacity utility function $\hat{J}^{(k,np)}(f, g) = 1 / J^{(k,np)}(f, g)$.

If an CNE exists for the game with the utility functions, then an CNE as well as a normalized equilibrium exist also for the game considering the cost functions.

We are now in the position to apply the following theorem to the game with utility functions.

**Theorem 3.2.** [2],[8]: Let $F$ be a closed, bounded and convex subset of $\mathbb{R}^m$ and for each $i \in I_p \cup I_{np}$ let the utility functional $\hat{J}^i : F \rightarrow \mathbb{R}$ be continuous in $F$ and either concave in $f^i$ for every $f^{-i}$ such that $f \in F$ if $i \in I_p$, or concave in $g^i$ for every $(f, g^{-i})$ such that $(f, g) \in F$ if $i \in I_{np}$. Then, the associated $(I+K)$-person nonzero-sum continuous-kernel game admits a Nash equilibrium solution.

Note that the constrained set $F$ defined by (3) with the properties $G$ and $G'$ for the cost functions meet the conditions of theorem 3.2. These conditions are met even by the priority and non-priority utility functions.
Theorem 3.3. Consider the cost functions of type-$G$ and $G'$. Then there exists a normalized Nash equilibrium for every specified vector $\vec{\alpha} > 0$ (componentwise) where $\vec{\alpha} = (\alpha^1, \alpha^2, ..., \alpha^{I+K})$.

4 CNE IN A PARALLEL LINKS TOPOLOGY

In this section, we investigate the problem of the Coupled Nash Equilibrium and Normalized Nash Equilibrium uniqueness for the non-cooperative game underlying our model. We present an example showing that in the proposed parallel link network model for routing with priority users and non-priority players the CNE may be not unique.

Example 4.1. Consider a network of two parallel links connecting a common source node to a common destination node. Link 1 has a capacity constraint of 1 and link 2 has capacity constraint of 6. There are two priority users each with a throughput demand $r^1 = r^2 = 1$, and one non-priority user with a throughput demand $r'$.

Let $J^{(i,p)}$, $i=1,2$ be the cost function of priority user $i$ such that $J^{(i,p)}(f) = \sum_{l=1}^{2} f^l_l T_l(f^l)$, and $J^{(k,np)}$, $k=1$ be the cost function of non-priority user $k$ such that $J^{(k,np)}(f,g) = \sum_{l=1}^{2} g^k_l T_l(f^l + g^l)$ where $T_1(f^1) = (4-f^1)^{-1}$, $T_2(f^2) = (4-f^2)^{-1} + 6$, $T_1(f^1+g^1) = (4-f^1-g^1)^{-1}$ and $T_2(f^2+g^2) = (4-f^2-g^2)^{-1} + 6$. Note that the cost functions are of type $A$ and $A'$.

One first Coupled Nash equilibrium is: For the first priority user: $f^1_1 = 1$, $f^2_1 = 0$, for the second priority user: $f^1_2 = 0$, $f^2_2 = 1$ and for the non-priority user: $g^1_1 = 0$, $g^2_1 = 1$. Another one is $f^1_1 = 0$, $f^1_2 = 1$ for the first prioriy user, $f^2_1 = 1$, $f^2_2 = 0$ for the second priory user and for the non-priory user $g^1_1 = 0$, $g^2_1 = 1$.

Proposition 4.2. In the proposed parallel link network model for routing with priority users and non-priority users the CNE may be not unique.

In the rest of this section, we consider the problem of CNE uniqueness and we present two particular cases.

Case 1: Absence of the side constraint.

Consider a network of parallel links connecting a common source node to a common destination node. Links have an infinite capacity. There are priority users and non-priority users.

It follows from our assumptions that the minimization in terms of cost function is equivalent to the following Kuhn-Tucker conditions:
For every \( i \in I_p \), there exists a Lagrange multipliers \( \lambda^{(i,p)} \) such that, for every link \( l \in \mathcal{L} \):
\[
K_l^{(i,p)}(f_l) = \lambda^{(i,p)} \text{ if } f^i_l > 0 \\
K_l^{(i,p)}(f_l) \geq \lambda^{(i,p)} \text{ if } f^i_l = 0
\]
The uniqueness of the NEP can be proved by using the following result:

**Theorem 4.3.** [7]: In a network of parallel links where the Links have an infinite capacity and the cost function \( J^{(i,p)}(f) \) of each user \( i \in I_p \) is of type \( A \), the NEP \( \hat{f} \) is unique.

**Proof.** see the proof of Theorem 1 in [7]. \( \square \)

**Case 2:** Presence of the side constraint [3].

In this case CNE may be not unique.

**Example 4.4.** Consider a network of two parallel links connecting a common source node to a common destination node. Link 1 has a capacity constraint of 1 and link 2 has capacity constraint of 10. There are two priority users (or two non-priority users), each with a throughput demand \( r_1 = r_2 = 1 \) between source node and destination. Let \( J^{(i,p)} \), \( i=1,2 \) be the cost function of user \( i \) such that \( J^{(i,p)}(f) = \sum_{l=1}^{2} f_l^i T_l(f_l) \) where \( T_1(f_1) = (2 - f_1)^{-1} \) and \( T_2(f_2) = (2 - f_2)^{-1} + 10 \) (\( T_l \) is the link costs of an M/M/1 queue), we note that the cost we chose are of type \( A \).

One first Nash equilibrium is: For the first user: \( f_1^1 = 1, f_1^2 = 0 \), for the second user: \( f_2^1 = 0, f_2^2 = 1 \). Another one is \( f_1^1 = 0, f_2^1 = 1 \) for the first user, and for the second user: \( f_1^2 = 1, f_2^2 = 0 \).

**Theorem 4.5.** Let the priority users cost functions be of type \( A \) and the non-priority users cost functions be of type \( A' \). Further Let \( (f,g) \) and \( (\hat{f},\hat{g}) \) to be two coupled Nash equilibria, and \( (\beta^{i,p}_l, \beta^{k,np}_l) \) and \( (\hat{\beta}^{i,p}_l, \hat{\beta}^{k,np}_l) \) be the corresponding Lagrange multipliers. Assume that for each link \( l \in \mathcal{L} \), \( \beta^{i,p}_l \leq \hat{\beta}^{i,p}_l \forall i \in I_p \) or \( \beta^{i,p}_l \geq \hat{\beta}^{i,p}_l \forall i \in I_p \). Then, \( f_l = \hat{f}_l \forall l \in \mathcal{L} \). Moreover if \( \beta^{k,np}_l \leq \hat{\beta}^{k,np}_l \) or \( \beta^{k,np}_l \geq \hat{\beta}^{k,np}_l \forall k \in I^{np} \) then \( g_l = \hat{g}_l \forall l \in \mathcal{L} \).

**Proof.** Let \( (f,g) \) and \( (\hat{f},\hat{g}) \) to be two coupled Nash equilibria. Then we have from (8) and (9):
\[
K_l^{(i,p)}(f_l^i, f_l^i) + \beta_l^{i,p} \geq \lambda^{(i,p)}; K_l^{(i,p)}(f_l^i, f_l^i) + \hat{\beta}_l^{i,p} = \lambda^{(i,p)} \text{ if } f^i_l > 0 \quad (10)
\]
\[
K_l^{(i,p)}(f_l^i, \hat{f}_l^i) + \hat{\beta}_l^{i,p} \geq \hat{\lambda}^{(i,p)}; K_l^{(i,p)}(f_l^i, \hat{f}_l^i) + \hat{\beta}_l^{i,p} = \hat{\lambda}^{(i,p)} \text{ if } f^i_l > 0 \quad (11)
\]
\[ K^{(k, np)}_l(g^k_l, g_l, f_l) + \beta^{(k, np)}_l \geq \lambda^{(k, np)}; K^{(k, np)}_l(g^k_l, g_l, f_l) + \beta^{(k, np)}_l = \lambda^{(k, np)} \text{ if } g^k_l > 0 \]

\[ K^{(k, np)}_l(g^k_l, g_l, f_l) + \beta^{(k, np)}_l \geq \hat{\lambda}^{(k, np)}; K^{(k, np)}_l(g^k_l, g_l, f_l) + \beta^{(k, np)}_l = \hat{\lambda}^{(k, np)} \text{ if } g^k_l > 0 \]

We use the same arguments as in proof of Theorem IV.1 in [3] to show that \( f_l = \hat{f}_l \forall l \in L \).

We now proceed to show that \( g_l = \hat{g}_l \forall l \in L \). We begin to show the following relations:

i) \( \{ \hat{\beta}^{k, np}_l < \beta^{k, np}_l, \hat{g}_l \geq g_l, \hat{f}_l = f_l \} \Rightarrow \hat{g}_l = g_l \), moreover if \( \hat{\lambda}^{k, np} \geq \lambda^{k, np} \) then \( \hat{g}^k_l \geq g^k_l \) and the last inequality is strict if \( \hat{g}^k_l > 0 \);

ii) \( \{ \beta^{k, np}_l > \hat{\beta}^{k, np}_l, \hat{g}_l \leq g_l, \hat{f}_l = f_l \} \Rightarrow \hat{g}_l = g_l \), moreover if \( \hat{\lambda}^{k, np} \leq \lambda^{k, np} \) then \( \hat{g}^k_l \leq g^k_l \) and the last inequality is strict if \( \hat{g}^k_l > 0 \);

iii) \( \{ \hat{\lambda}^{k, np} \leq \lambda^{k, np}, \hat{\beta}^{k, np}_l \geq \beta^{k, np}_l, \hat{g}_l \geq g_l, \hat{f}_l = f_l \} \Rightarrow \hat{g}^k_l \leq g^k_l \);

iv) \( \{ \hat{\lambda}^{k, np} \geq \lambda^{k, np}, \hat{\beta}^{k, np}_l \leq \beta^{k, np}_l, \hat{g}_l \leq g_l, \hat{f}_l = f_l \} \Rightarrow \hat{g}^k_l \geq g^k_l \).

We will show only i) and iii), since ii) and iv) can be derived by symmetry.

Assume that \( \hat{\beta}^{k, np}_l < \beta^{k, np}_l \) and \( \hat{g}_l \geq g_l \). Note that the last inequality implies that:

\[ G'(\hat{g}_l) \geq G'(g_l) \]  

(14)

Since \( \hat{\beta}^{k, np}_l < \beta^{k, np}_l \) then from (9) \( G'(g_l) = 0 \) and from (7) \( G'(\hat{g}_l) \leq 0 \). Thus \( G'(g_l) \geq G'(\hat{g}_l) \) and it follows by (14) that \( G'(g_l) = G'(\hat{g}_l) \) and \( g_l = \hat{g}_l \). Moreover if \( \hat{\lambda}^{k, np} \geq \lambda^{k, np} \) and \( \hat{f}_l = f_l \) then (i) holds trivially if \( \hat{g}^k_l = 0 \). Otherwise, if \( \hat{g}^k_l > 0 \), then (12) and (13) together with our assumptions imply that:

\[ K^{(k, np)}_l(g^k_l, g_l, f_l) + \beta^{(k, np)}_l = \lambda^{(k, np)} \leq \hat{\lambda}^{(k, np)} \leq K^{(k, np)}_l(g^k_l, \hat{g}_l, \hat{f}_l) + \beta^{(k, np)}_l \nonumber \]

\[ < K^{(k, np)}_l(\hat{g}^k_l, g_l, f_l) + \beta^{(k, np)}_l \]  

(15)

where the last inequality follows from the monotonicity of \( K^{(k, np)}_l \) in its second argument, \( \hat{f}_l = f_l \) and \( \hat{\beta}^{k, np}_l < \beta^{k, np}_l \). Thus \( K^{(k, np)}_l(\hat{g}^k_l,g_l,f_l) \leq K^{(k, np)}_l(\hat{g}^k_l,g_l,f_l) \). Now, since \( K^{(k, np)}_l \) is nondecreasing in its first argument, then \( \hat{g}^k_l > g^k_l \). This establishes (i).

Now, we assume that \( \hat{\lambda}^{k, np} \leq \lambda^{k, np}, \hat{\beta}^{k, np}_l \geq \beta^{k, np}_l, \hat{g}_l \geq g_l \) and \( \hat{f}_l = f_l \). Note that (iii) holds trivially if \( \hat{g}^k_l = 0 \). Otherwise, if \( \hat{g}^k_l > 0 \), then (12) and (13) together with our assumptions imply that:

\[ K^{(k, np)}_l(\hat{g}^k_l, \hat{g}_l, \hat{f}_l) + \hat{\beta}^{(k, np)}_l = \hat{\lambda}^{(k, np)} \leq \lambda^{(k, np)} \leq \]  

\[ K^{(k, np)}_l(g^k_l, g_l, f_l) + \beta^{(k, np)}_l \leq K^{(k, np)}_l(g^k_l, \hat{g}_l, \hat{f}_l) + \beta^{(k, np)}_l \]  

(16)
where the last inequality follows from monotonicity of $K^{(k,np)}_l$ in its second argument, $\hat{f}_i = f_i$ and $\hat{\beta}^{(k,np)}_l \geq \beta^{(k,np)}_l$. Thus $K^{(k,np)}_l (\hat{g}^k_l, \hat{g}_l, \hat{f}_i) = K^{(k,np)}_l (g^k_l, \hat{g}_l, f_i)$. Now, since $K^{(k,np)}_l$ is nondecreasing in its first argument, then $g^k_l \leq g^k_\ell$. This establishes (iii).

We consider the following notation: $\mathcal{L}_1 = \{l : \hat{g}_l > g_l; \hat{f}_i = f_i\}$, $I^{np}_1 = \{k : \hat{\lambda}^{(k,np)} > \lambda^{(k,np)}\}$, $\mathcal{L}_2 = \{l : \hat{g}_l \leq g_l; \hat{\beta}^{(k,np)}_l \leq \beta^{(k,np)}_l; \hat{f}_i = f_i\}$ and $\mathcal{L}_3 = \{l : \hat{g}_l \leq g_l; \hat{\beta}^{(k,np)}_l > \beta^{(k,np)}_l; \hat{f}_i = f_i\}$. Note that $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$. Assume that $\mathcal{L}_1$ is nonempty; then it follows by (iv) that for $k \in I^{np}_1$: $\sum_{l \in \mathcal{L}_1} \hat{g}^k_l = r^k - \sum_{l \in \mathcal{L}_2} \hat{g}^k_l - \sum_{l \in \mathcal{L}_3} \hat{g}^k_l \leq r^k - \sum_{l \in \mathcal{L}_2} g^k_l - \sum_{l \in \mathcal{L}_3} g^k_l \leq \sum_{l \in \mathcal{L}_1} g^k_l + \sum_{l \in \mathcal{L}_2} (g^k_l - \hat{g}^k_l) + \sum_{l \in \mathcal{L}_3} (g^k_l - \hat{g}^k_l)$. Note that (i) implies that $\{l \in \mathcal{L}_1 : \beta^{(k,np)}_l < \beta^{(k,np)}_l\} = \emptyset$, and it follows that if $l \in \mathcal{L}_1$, $k \notin I^{np}_1$, and from (iii) $\hat{g}^k_l \leq g^k_l$. Therefore,

$$
\sum_{l \in \mathcal{L}_1} \hat{g}_l = \sum_{l \in \mathcal{L}_1} \sum_{k \in I^{np}_1} \hat{g}^k_l + \sum_{l \in \mathcal{L}_1} \sum_{k \notin I^{np}_1} \hat{g}^k_l \\
\leq \sum_{l \in \mathcal{L}_1} \sum_{k \in I^{np}_1} g^k_l + \sum_{l \in \mathcal{L}_1} \sum_{k \in I^{np}_1} (g^k_l - \hat{g}^k_l) + \sum_{l \in \mathcal{L}_1} \sum_{k \notin I^{np}_1} g^k_l \\
= \sum_{l \in \mathcal{L}_1} g_l + \sum_{l \in \mathcal{L}_1} \sum_{k \in I^{np}_1} (g^k_l - \hat{g}^k_l) \\
= \sum_{l \in \mathcal{L}_1} g_l + \sum_{l \in \mathcal{L}_1} (g_l - \hat{g}_l) - \sum_{l \in \mathcal{L}_1} \sum_{k \notin I^{np}_1} (g^k_l - \hat{g}^k_l) \\
\leq \sum_{l \in \mathcal{L}_1} g_l.
$$

The last inequality follows from (ii), since for $l \in \mathcal{L}_3$, $\hat{g}_l = g_l$ and for $l \in \mathcal{L}_3$ and $k \notin I^{np}_1$, $g^k_l \geq \hat{g}^k_l$.

Inequality (17) and the definition of $\mathcal{L}_1$ are contradictory, which implies that $\mathcal{L}_1$ is an empty set. By symmetry, it can be derived that the set $\{l : \hat{g}_l < g_l; \hat{f}_i = f_i\}$ is empty. Thus we conclude that, $\hat{g}_l = g_l$, $\forall l \in \mathcal{L}$. ∎

## 5 NORMALIZED NASH EQUILIBRIUM

### 5.1 Uniqueness of the Normalized Nash Equilibrium

The following result shows that the parallel links network has a unique normalized Nash equilibrium for any $\alpha > 0$.

**Theorem 5.1.** In a network of parallel links where the priority users cost functions are of type A and the non-priority users cost functions are of type A', the normalized Nash equilibrium for any $\alpha > 0$ is unique.
Proof. Assumptions of Theorem 4.5 hold in this case, and hence we have the uniqueness for link utilizations by the two typologies of users: priority and non-priority users. Then the theorem follows directly from Thm. 4.1 in [3].

5.2 Properties of Normalized Nash Equilibrium

We assume that the priority users costs are of type $A$, the non-priority users costs are of type $A'$ and $K^{i,(p)}_l = K^{p}_l$; $K^{(k,np)}_l = K^{np}_l$ and $\alpha^j = \alpha, \forall j \in I^p \cup I^{np}$ where $\alpha$ is some positive number.

**Lemma 5.2.** Assume that $f^i_l > f^i_{\hat{l}}$ and $g^k_l > g^k_{\hat{k}}$ hold for some link $\hat{l}$, priority users $i$ and $\hat{i}$, and non-priority users $k$ and $\hat{k}$. Then $f^i_l \geq f^i_{\hat{l}}$ and $g^k_l \geq g^k_{\hat{k}}$ for all $l \in \mathcal{L}$; moreover, the last inequalities are strict respectively if $f^i_l > 0$ and $g^k_l > 0$.

**Proof.** We assume that $\alpha^j = 1, \forall j \in I^p \cup I^{np}$ and we choose an arbitrary link $l$.

- If $f^i_l = 0$ and $g^k_l = 0$ then the result is trivial.
- If $f^i_l > 0$ and $g^k_l > 0$ then $f^i_l \geq f^i_{\hat{l}}$ is trivial, and from Kuhn-Tucker conditions we have that:
  \[
  \lambda^{k,np} = \beta^{np}_l + K^{np}_l(g^k_l, g_i, f_l) \leq \beta^{np}_l + K^{np}_l(g^k_{\hat{l}}, g_i, f_{\hat{l}}) \text{ and, since } g^k_l > g^k_{\hat{l}} \text{ implies } g^k_l > 0, \text{ we have } \lambda^{k,np} = \beta^{np}_l + K^{np}_l(g^k_l, g_i, f_l) \leq \beta^{np}_l + K^{np}_l(g^k_l, g_i, f_l).
  \]
  Thus, we have $\beta^{np}_l + K^{np}_l(g^k_l, g_i, f_l) \leq \beta^{np}_l + K^{np}_l(g^k_{\hat{l}}, g_i, f_{\hat{l}}) < \beta^{np}_l + K^{np}_l(g^k_{\hat{l}}, g_i, f_{\hat{l}})$, i.e., $K^{np}_l(g^k_l, g_i, f_l) < K^{np}_l(g^k_{\hat{l}}, g_i, f_{\hat{l}})$ which implies $g^k_l < g^k_{\hat{l}}$.
- If $f^i_l > 0$ and $g^k_l = 0$ then $g^k_l \geq g^k_{\hat{k}}$ is trivial, and from Kuhn-Tucker conditions we have that:
  \[
  \lambda^{i,p} = \beta^{p}_l + K^{p}_l(f^i_l, f_i, g_i) \leq \beta^{p}_l + K^{p}_l(f^i_{\hat{l}}, f_i, g_i) \text{ and, since } f^i_l > f^i_{\hat{l}} \text{ implies } f^i_l > 0, \text{ we have } \lambda^{i,p} = \beta^{p}_l + K^{p}_l(f^i_l, f_i, g_i) \leq \beta^{p}_l + K^{p}_l(f^i_l, f_i, g_i).
  \]
  Thus, we have $\beta^{p}_l + K^{p}_l(f^i_l, f_i, g_i) \leq \beta^{p}_l + K^{p}_l(f^i_{\hat{l}}, f_i, g_i) < \beta^{p}_l + K^{p}_l(f^i_{\hat{l}}, f_i, g_i) \leq \beta^{p}_l + K^{p}_l(f^i_l, f_i, g_i)$, i.e., $K^{p}_l(f^i_l, f_i, g_i) < K^{p}_l(f^i_{\hat{l}}, f_i, g_i)$ which implies $f^i_l < f^i_{\hat{l}}$.
- If $f^i_l > 0$ and $g^k_l > 0$ then:
  From Kuhn-Tucker conditions we have that:
  \[
  \lambda^{i,p} = \beta^{p}_l + K^{p}_l(f^i_l, f_i, g_i) \leq \beta^{p}_l + K^{p}_l(f^i_{\hat{l}}, f_i, g_i) \text{ and, since } f^i_l > f^i_{\hat{l}} \text{ implies } f^i_l > 0, \text{ we have } \lambda^{i,p} = \beta^{p}_l + K^{p}_l(f^i_l, f_i, g_i) \leq \beta^{p}_l + K^{p}_l(f^i_l, f_i, g_i).
  \]
Thus, we have \( \beta_i^p + K_i^p(f_i^i, f_i, g_i) \leq \beta_i^p + K_i^p(f_i^j, f_i, g_i) \leq \beta_i^p + K_i^p(f_i^j, f_i, g_i) \), i.e., \( K_i^p(f_i^i, f_i, g_i) < K_i^p(f_i^j, f_i, g_i) \) which implies \( f_i^j < f_i^i \).

And:
\[
\lambda^{(k, np)} = \beta_i^{np} + K_i^{np}(g_i^k, g_i, f_i) \leq \beta_i^{np} + K_i^{np}(g_i^k, g_i, f_i) \text{ and, since } g_i^k > g_i^k \text{ implies } g_i^k > 0, \text{ we have } \lambda^{(k, np)} = \beta_i^{np} + K_i^{np}(g_i^k, g_i, f_i) \leq \beta_i^{np} + K_i^{np}(g_i^k, g_i, f_i).\]

Thus, we have \( \beta_i^{np} + K_i^{np}(g_i^k, g_i, f_i) \leq \beta_i^{np} + K_i^{np}(g_i^k, g_i, f_i) < \beta_i^{np} + K_i^{np}(g_i^k, g_i, f_i) \), i.e., \( K_i^{np}(g_i^k, g_i, f_i) < K_i^{np}(g_i^k, g_i, f_i) \) which implies \( g_i^k < g_i^k \).

\[ \Box \]

**Theorem 5.3.** Consider the identical type \( A \) priority cost functions and the identical type \( A' \) non-priority cost functions. The following relations hold for all link \( l \in \mathcal{L} \):

\[
\begin{align*}
    r^i > r^i & \Rightarrow f_i^i \geq f_i^i \\
    r^k > r^k & \Rightarrow g_i^k \geq g_i^k \\
    r^i = r^i & \Rightarrow f_i^i = f_i^i \\
    r^k = r^k & \Rightarrow g_i^k = g_i^k
\end{align*}
\]

*Proof.* We show that \( f_i^i \geq f_i^i \) for all links \( l \in \mathcal{L} \). Assume that \( f_i^i < f_i^i \) for some \( \hat{l} \). Then, by the Lemma 5.2 we have \( f_i^i \leq f_i^i \) on all other links, which upon summation yields \( r^i < r^i \), which contradicts \( r^i > r^i \).

A similar approach may be used to prove \( g_i^k \geq g_i^k \) if \( r^k > r^k \) for all link \( l \in \mathcal{L} \). \[ \Box \]

## 6 CONCLUSION

In this paper we have presented a game theoretic non cooperative model for the QoS routing in communication networks shared by two types of players: priority users and non-priority users.

We have established that for priority and non-priority users cost functions, the NEP of the game underlying our routing model exists. We have shown that the extra constraints may cause multiple equilibria for scenarios in which a single equilibrium would exist in their absence. We then advocated the use of the more refined equilibrium concept of normalized Nash equilibrium. We further showed that it is unique in the parallel topology.

In works future, we plan to extend our results to general topology and other forms of constraints.
References


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