Compatibility of Type ($\beta$) and Fixed Point Theorem in Fuzzy Metric Space

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Abstract

In this paper, the concept of compatible maps of type ($\beta$) in fuzzy metric space has been applied to prove common fixed point theorem. A fixed point theorem for six self maps has been established using the concept of compatible maps of type ($\beta$), which generalizes the result of Cho [1].

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1. Introduction

The concept of Fuzzy sets was initially investigated by Zadeh [13] as a new way to represent vagueness in everyday life. Subsequently, it was developed by many authors and used in various fields. To use this concept in Topology and Analysis, several researchers have defined Fuzzy metric space in various ways. In this paper we deal with the Fuzzy metric space defined by Kramosil and Michalek
[10] and modified by George and Veeramani [4]. Recently, Grebiec [5] has proved fixed point results for Fuzzy metric space. In the sequel, Singh and Chauhan [12] introduced the concept of compatible mappings of Fuzzy metric space and proved the common fixed point theorem. Jungck et al. [8] introduced the concept of compatible maps of type (A) in metric space and proved fixed point theorems. Cho [2, 3] introduced the concept of compatible maps of type (α) and compatible maps of type (β) in fuzzy metric space. Using the concept of compatible maps of type (A), Jain et al. [6] proved a fixed point theorem for six self maps in a fuzzy metric space. Using the concept of compatible maps of type (β), Jain et al. [7] proved a fixed point theorem in fuzzy metric space.

In this paper, a fixed point theorem for six self maps has been established using the concept of compatible maps of type (β) and weak compatible maps, which generalizes the result of Cho [1].

For the sake of completeness, we recall some definitions and known results in Fuzzy metric space.

2. Preliminaries

Definition 2.1. [11] A binary operation * : [0, 1] × [0, 1] → [0, 1] is called a t-norm if ([0, 1], *) is an abelian topological monoid with unit 1 such that a * b ≤ c * d whenever a ≤ c and b ≤ d for a, b, c, d ∈ [0, 1].

Examples of t-norms are a * b = ab and a * b = min{a, b}.

Definition 2.2. [11] The 3-tuple (X, M, *) is said to be a Fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a Fuzzy set in X^2 × [0, ∞) satisfying the following conditions:

for all x, y, z ∈ X and s, t > 0.

(FM-1) M(x, y, 0) = 0,

(FM-2) M(x, y, t) = 1 for all t > 0 if and only if x = y,

(FM-3) M(x, y, t) = M(y, x, t),

(FM-4) M(x, y, t) * M(y, z, s) ≤ M(x, z, t + s),

(FM-5) M(x, y, .) : [0, ∞) → [0, 1] is left continuous,

(FM-6) \lim_{t \to \infty} M(x, y, t) = 1.
Note that $M(x, y, t)$ can be considered as the degree of nearness between $x$ and $y$ with respect to $t$. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$. The following example shows that every metric space induces a Fuzzy metric space.

**Example 2.1.** [11] Let $(X, d)$ be a metric space. Define $a * b = \min \{a, b\}$ and 
$$M(x, y, t) = \frac{t}{t + d(x, y)}$$
for all $x, y \in X$ and all $t > 0$. Then $(X, M, *)$ is a Fuzzy metric space. It is called the Fuzzy metric space induced by $d$.

**Definition 2.3.** [11] A sequence $\{x_n\}$ in a Fuzzy metric space $(X, M, *)$ is said to be a *Cauchy sequence* if and only if for each $\varepsilon > 0$, $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.

The sequence $\{x_n\}$ is said to converge to a point $x$ in $X$ if and only if for each $\varepsilon > 0$, $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - \varepsilon$ for all $n \geq n_0$.

A Fuzzy metric space $(X, M, *)$ is said to be *complete* if every Cauchy sequence in it converges to a point in it.

**Definition 2.4.** [12] Self mappings $A$ and $S$ of a Fuzzy metric space $(X, M, *)$ are said to be *compatible* if and only if $M(ASx_n, SAx_n, t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in $X$ such that $Sx_n, Ax_n \to p$ for some $p$ in $X$ as $n \to \infty$.

**Definition 2.5.** [3] Self maps $A$ and $S$ of a Fuzzy metric space $(X, M, *)$ are said to be compatible maps of type $(\beta)$ if $M(AAx_n, SSx_n, t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in $X$ such that $Sx_n, Ax_n \to p$ for some $p$ in $X$ as $n \to \infty$.

**Definition 2.6.** [7] Two maps $A$ and $B$ from a Fuzzy metric space $(X, M, *)$ into itself are said to be *weakly compatible* if they commute at their coincidence points, i.e. $Ax = Bx$ implies $ABx = BAx$ for some $x \in X$.

**Remark 2.1.** [7] The concept of compatible maps of type $(\beta)$ and weak compatibility is more general than the concept of compatible maps in a Fuzzy metric space.

**Proposition 2.1.** [6] In a Fuzzy metric space $(X, M, *)$ limit of a sequence is unique.

**Lemma 2.1.** [5] Let $(X, M, *)$ be a fuzzy metric space. Then for all $x, y \in X$, $M(x, y, \cdot)$ is a non-decreasing function.

**Lemma 2.2.** [1] Let $(X, M, *)$ be a fuzzy metric space. If there exists $k \in (0, 1)$ such that for all $x, y \in X$, $M(x, y, kt) \geq M(x, y, t) \forall \ t > 0$, then $x = y$. 
Lemma 2.3. [6] Let \( \{x_n\} \) be a sequence in a fuzzy metric space \((X, M, *)\). If there exists a number \( k \in (0, 1) \) such that
\[
M(x_{n+2}, x_{n+1}, kt) \geq M(x_{n+1}, x_n, t) \quad \forall \ t > 0 \text{ and } n \in \mathbb{N}.
\]
Then \( \{x_n\} \) is a Cauchy sequence in \( X \).

Lemma 2.4. [9] The only t-norm \(*\) satisfying \( r * r \geq r \) for all \( r \in [0, 1] \) is the minimum t-norm, that is
\[
a * b = \min \{a, b\} \text{ for all } a, b \in [0, 1].
\]

3. Main Result

Theorem 3.1. Let \((X, M, *)\) be a complete fuzzy metric space and let \(A, B, S, T, P\) and \(Q\) be mappings from \(X\) into itself such that the following conditions are satisfied:

(a) \( P(X) \subseteq ST(X) \), \( Q(X) \subseteq AB(X) \);
(b) \( AB = BA \), \( ST = TS \), \( PB = BP \), \( QT = TQ \);
(c) either \( P \) or \( AB \) is continuous;
(d) \((P, AB)\) is compatible of type \((\beta)\) and \((Q, ST)\) is weak-compatible;
(e) there exists \( q \in (0, 1) \) such that for every \( x, y \in X \) and \( t > 0 \)
\[
M(Px, Qy, qt) \geq M(ABx, STy, t) * M(Px, ABx, t) * M(Qy, STy, t) * M(Px, STy, t).
\]

Then \( A, B, S, T, P \) and \( Q \) have a unique common fixed point in \( X \).

Proof: Let \( x_0 \in X \). From (a) there exist \( x_1, x_2 \in X \) such that
\[
P x_0 = STx_1 \text{ and } Qx_1 = ABx_2.
\]
Inductively, we can construct sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that
\[
Px_{2n-2} = STx_{2n-1} = y_{2n-1} \text{ and } Qx_{2n-1} = ABx_{2n} = y_{2n} \quad \text{for } n = 1, 2, 3, \ldots.
\]

Step 1. Put \( x = x_{2n} \) and \( y = x_{2n+1} \) in (e), we get
\[
M(Px_{2n}, Qx_{2n+1}, qt) \geq M(ABx_{2n}, STx_{2n+1}, t) * M(Px_{2n}, ABx_{2n}, t) * M(Qx_{2n+1}, STx_{2n+1}, t).
\]

From lemma 2.1 and 2.2, we have
\[
M(y_{2n}, y_{2n+1}, qt) \geq M(y_{2n}, y_{2n+1}, t).
\]
Similarly, we have
\[ M(y_{2n+2}, y_{2n+3}, qt) \geq M(y_{2n+1}, y_{2n+2}, t). \]
Thus, we have
\[ M(y_{n+1}, y_{n+2}, qt) \geq M(y_n, y_{n+1}, t) \text{ for } n = 1, 2, \ldots \]
\[ M(y_n, y_{n+1}, t) \geq M(y_n, y_{n+1}, t/q) \]
\[ \geq M(y_{n-2}, y_{n-1}, t/q^2) \]
\[ \geq \ldots \]
\[ \geq M(y_1, y_2, t/q^n) \rightarrow 1 \text{ as } n \rightarrow \infty, \]
and hence \[ M(y_n, y_{n+1}, t) \rightarrow 1 \text{ as } n \rightarrow \infty \text{ for any } t > 0. \]

For each \( \varepsilon > 0 \) and \( t > 0 \), we can choose \( n_0 \in \mathbb{N} \) such that
\[ M(y_n, y_{n+1}, t) > 1 - \varepsilon \text{ for all } n > n_0. \]

For \( m, n \in \mathbb{N} \), we suppose \( m \geq n \). Then we have
\[ M(y_n, y_m, t) \geq M(y_n, y_{n+1}, t/m-n) \]
\[ \times \ldots \times M(y_{m-1}, y_m, t/m-n) \]
\[ \geq (1 - \varepsilon)^{m-n} \]
and hence \( \{y_n\} \) is a Cauchy sequence in \( X \).

Since \((X, M, \ast)\) is complete, \( \{y_n\} \) converges to some point \( z \in X \). Also its subsequences converges to the same point i.e. \( z \in X \)
i.e.,
\[ \{Qx_{2n+1}\} \rightarrow z \quad \text{and} \quad \{STx_{2n+1}\} \rightarrow z \quad (1) \]
\[ \{Px_{2n}\} \rightarrow z \quad \text{and} \quad \{ABx_{2n}\} \rightarrow z. \quad (2) \]

**Case I.** Suppose \( AB \) is continuous.

Since \( AB \) is continuous, we have
\[ (AB)^2x_{2n} \rightarrow ABz \quad \text{and} \]
\[ ABPx_{2n} \rightarrow ABz. \]

As \((P, AB)\) is compatible pair of type \((\beta)\), we have
M(PP_{x_{2n}}, (AB)(AB)x_{2n}, t) = 1, for all t > 0

or,

M(PP_{x_{2n}}, ABz, t) = 1.

Therefore,

PP_{x_{2n}} \rightarrow ABz.

**Step 2.** Put \( x = ABx_{2n} \) and \( y = x_{2n+1} \) in (e), we get

\[
M(PABx_{2n}, Qx_{2n+1}, qt) \geq M(ABABx_{2n}, STx_{2n+1}, t) \ast M(PABx_{2n}, ABABx_{2n}, t) \ast M(Qx_{2n+1}, STx_{2n+1}, t) \ast M(PABx_{2n}, STx_{2n+1}, t).
\]

Taking \( n \rightarrow \infty \), we get

\[
M(ABz, z, qt) \geq M(ABz, z, t) \ast M(ABz, ABz, t) \ast M(z, z, t) \ast M(ABz, z, t) \geq M(ABz, z, t) \ast M(ABz, z, t)
\]

i.e. \( M(ABz, z, qt) \geq M(ABz, z, t) \).

Therefore, by using lemma 2.2, we get

\[
ABz = z.
\]  (3)

**Step 3.** Put \( x = z \) and \( y = x_{2n+1} \) in (e), we have

\[
M(Pz, Qx_{2n+1}, qt) \geq M(ABz, STx_{2n+1}, t) \ast M(Pz, ABz, t) \ast M(Qx_{2n+1}, STx_{2n+1}, t) \ast M(Pz, STx_{2n+1}, t).
\]

Taking \( n \rightarrow \infty \) and using equation (1), we get

\[
M(Pz, z, qt) \geq M(z, z, t) \ast M(Pz, z, t) \ast M(z, z, t) \ast M(Pz, z, t) \geq M(Pz, z, t) \ast M(Pz, z, t)
\]

i.e. \( M(Pz, z, qt) \geq M(Pz, z, t) \).

Therefore, by using lemma 2.2, we get

\[
Pz = z.
\]

Therefore, \( ABz = Pz = z \).

**Step 4.** Putting \( x = Bz \) and \( y = x_{2n+1} \) in condition (e), we get

\[
M(PBz, Qx_{2n+1}, qt) \geq M(ABBz, STx_{2n+1}, t) \ast M(PBz, ABBz, t) \ast M(Qx_{2n+1}, STx_{2n+1}, t) \ast M(PBz, STx_{2n+1}, t).
\]
As $BP = PB, AB = BA$, so we have

\[
P(Bz) = B(Pz) = Bz \quad \text{and} \quad (AB)(Bz) = (BA)(Bz) = B(ABz) = Bz.
\]

Taking $n \to \infty$ and using (1), we get

\[
M(Bz, z, qt) \geq M(Bz, z, t) \ast M(Bz, Bz, t) \ast M(z, z, t) \ast M(Bz, z, t)
\]
\[
\geq M(Bz, z, t) \ast M(Bz, z, t)
\]

i.e. $M(Bz, z, qt) \geq M(Bz, z, t)$.

Therefore, by using lemma 2.2, we get

\[
Bz = z
\]

and also we have

\[
ABz = z
\]

\[
\Rightarrow Az = z.
\]

Therefore, $Az = Bz = Pz = z$. \hfill (4)

**Step 5.** As $P(X) \subset ST(X)$, there exists $u \in X$ such that

\[
z = Pz = STu.
\]

Putting $x = x_{2n}$ and $y = u$ in (e), we get

\[
M(Px_{2n}, Qu, qt) \geq M(ABx_{2n}, STu, t) \ast M(Px_{2n}, ABx_{2n}, t)
\]
\[
\ast M(Qu, STu, t) \ast M(Px_{2n}, STu, t).
\]

Taking $n \to \infty$ and using (1) and (2), we get

\[
M(z, Qu, qt) \geq M(z, z, t) \ast M(z, z, t) \ast M(Qu, z, t) \ast M(z, z, t)
\]
\[
\geq M(Qu, z, t)
\]

i.e. $M(z, Qu, qt) \geq M(z, Qu, t)$.

Therefore, by using lemma 2.2, we get

\[
Qu = z.
\]

Hence $STu = z = Qu$.

Since $(Q, ST)$ is weak compatible, therefore, we have
QSTu = STQu.
Thus Qz = STz.

Step 6. Putting x = x_{2n} and y = z in (e), we get
\[ M(Px_{2n}, Qz, qt) \geq M(ABx_{2n}, STz, t) * M(Px_{2n}, ABx_{2n}, t) \]
\[ * M(Qz, STz, t) * M(Px_{2n}, STz, t). \]
Taking n \to \infty and using (2) and step 5, we get
\[ M(z, Qz, qt) \geq M(z, Qz, t) * M(z, z, t) * M(Qz, Qz, t) * M(z, Qz, t) \]
\[ \geq M(z, Qz, t) * M(z, Qz, t) \]
i.e. \[ M(z, Qz, qt) \geq M(z, Qz, t). \]
Therefore, by using lemma 2.2, we get
Qz = z.

Step 7. Putting x = x_{2n} and y = Tz in (e), we get
\[ M(Px_{2n}, QTz, qt) \geq M(ABx_{2n}, STTz, t) * M(Px_{2n}, ABx_{2n}, t) \]
\[ * M(QTz, STTz, t) * M(Px_{2n}, STTz, t). \]
As QT = TQ and ST = TS, we have
QTz = TQz = Tz and
ST(Tz) = T(STz) = TQz = Tz.
Taking n \to \infty, we get
\[ M(z, Tz, qt) \geq M(z, Tz, t) * M(z, z, t) * M(Tz, Tz, t) * M(z, Tz, t) \]
\[ \geq M(z, Tz, t) * M(z, Tz, t) \]
i.e. \[ M(z, Tz, qt) \geq M(z, Tz, t). \]
Therefore, by using lemma 2.2, we get
Tz = z.
Now STz = Tz = z implies Sz = z.
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Hence \( Sz = Tz = Qz = z \). \hfill (5)

Combining (4) and (5), we get

\[
Az = Bz = Pz = Qz = Tz =Sz = z.
\]

Hence, \( z \) is the common fixed point of \( A, B, S, T, P \) and \( Q \).

**Case II.** Suppose \( P \) is continuous.

As \( P \) is continuous,

\[
P^{2n}x_{2n} \to Pz \quad \text{and} \quad P(AB)x_{2n} \to Pz.
\]

As \((P, AB)\) is compatible pair of type \((\beta)\),

\[
M(PPx_{2n}, (AB)(AB)x_{2n}, t) = 1, \text{ for all } t > 0
\]

or,

\[
M(Pz, (AB)(AB)x_{2n}, t) = 1.
\]

Therefore, \((AB)^2x_{2n} \to Pz\).

**Step 8.** Putting \( x = Px_{2n} \) and \( y = x_{2n+1} \) in condition (e), we have

\[
M(PPx_{2n}, Qx_{2n+1}, qt) \geq M(ABPx_{2n}, STx_{2n+1}, t) * M(PPx_{2n}, ABPx_{2n}, t)
\]

\[
* M(Qx_{2n+1}, STx_{2n+1}, t) * M(PPx_{2n}, STx_{2n+1}, t).
\]

Taking \( n \to \infty \), we get

\[
M(Pz, z, qt) \geq M(Pz, z, t) * M(Pz, Pz, t) \geq M(Pz, z, t).
\]

i.e.

\[
M(Pz, z, qt) \geq M(Pz, z, t).
\]

Therefore by using lemma 2.2, we have

\[
Pz = z.
\]

**Step 9.** Put \( x = ABx_{2n} \) and \( y = x_{2n+1} \) in (e), we get

\[
M(PABx_{2n}, Qx_{2n+1}, qt) \geq M(ABABx_{2n}, STx_{2n+1}, t) * M(PABx_{2n}, ABABx_{2n}, t)
\]

\[
* M(Qx_{2n+1}, STx_{2n+1}, t) * M(PABx_{2n}, STx_{2n+1}, t).
\]

Taking \( n \to \infty \), we get

\[
M(ABz, z, qt) \geq M(ABz, z, t) * M(ABz, ABz, t) \geq M(ABz, z, t).
\]

Therefore by using lemma 2.2, we have

\[
Pz = z.
\]
i.e. \( M(ABz, z, qt) \geq M(ABz, z, t) \).

Therefore, by using lemma 2.2, we get

\[ ABz = z. \]

Therefore,

\[ ABz = z = Pz. \]

Now, apply step 4, to get \( Bz = z \) and so

\[ Az = Bz = Pz = z. \]

Further, using steps 5, 6, 7, we get

\[ Qz = STz =Sz = Tz = z \]

i.e. \( z \) is the common fixed point of the six maps \( A, B, S, T, P \) and \( Q \) in this case also.

**Uniqueness**: Let \( u \) be another common fixed point of \( A, B, S, T, P \) and \( Q \).

Then \( Au = Bu = Pu = Qu = Su = Tu = u \).

Put \( x = z \) and \( y = u \) in (e), we get

\[
M(Pz, Qu, qt) \geq M(ABz, STu, t) \ast M(Pz, ABz, t) \ast M(Qu, STu, t) \ast M(Pz, STu, t).
\]

Taking \( n \to \infty \), we get

\[
M(z, u, qt) \geq M(z, u, t) \ast M(z, z, t) \ast M(u, u, t) \geq M(z, u, t).
\]

Therefore by using lemma (2.2), we get

\[ z = u. \]

Therefore \( z \) is the unique common fixed point of self maps \( A, B, S, T, P \) and \( Q \).

**Remark 3.1.** If we take \( B = T = I \), the identity map on \( X \) in theorem 3.1, then condition (b) is satisfied trivially and we get

**Corollary 3.1.** Let \( (X, M, *) \) be a complete fuzzy metric space and let \( A, S, P \) and \( Q \) be mappings from \( X \) into itself such that the following conditions are satisfied:
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(a) $P(X) \subset S(X)$, $Q(X) \subset A(X)$;
(b) either $A$ or $P$ is continuous;
(c) $(P, A)$ is compatible of type ($\beta$) and $(Q, S)$ is weak-compatible;
(d) there exists $q \in (0, 1)$ such that for every $x, y \in X$ and $t > 0$

\[ M(Px, Qy, qt) \geq M(Ax, Sy, t) \times M(Px, Ax, t) \times M(Qy, Sy, t) \times M(Px, Sy, t). \]

Then $A$, $S$, $P$ and $Q$ have a unique common fixed point in $X$.

**Remark 3.2.** In view of remark 3.1, corollary 3.1 is a generalization of the result of Cho [1] in the sense that condition of compatibility of the pairs of self maps has been restricted to compatibility of type ($\beta$) and weak compatibility and only one map of the first pair is needed to be continuous.

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