Direct Method and Stability of Additive Functional Equation in NAN-Spaces


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Abstract

In this paper, using direct method we prove the Hyers-Ulam-Rassias stability of an additive functional equation in non-Archimedean normed spaces.

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1 Introduction and preliminaries

A valuation is a function $|\cdot|$ from a field $K$ into $[0, \infty)$ such that, for all $r, s \in K$, the following conditions hold: (a) $|r| = 0$ if and only if $r = 0$; (b) $|rs| = |r||s|$;
A field $K$ is called a \textit{valued field} if $K$ carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by
\[ |r + s| \leq \max\{|r|, |s|\} \]
for all $r, s \in K$, then the function $|\cdot|$ is called a \textit{non-Archimedean valuation} and the field is called a \textit{non-Archimedean field}. Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

\textbf{Definition 1.1} Let $X$ be a vector space over a field $K$ with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \to [0, \infty)$ is called a non-Archimedean norm if the following conditions hold:

(a) $\|x\| = 0$ if and only if $x = 0$ for all $x \in X$;

(b) $\|rx\| = |r|\|x\|$ for all $r \in K$ and $x \in X$; (c) the strong triangle inequality holds:
\[ \|x + y\| \leq \max\{\|x\|, \|y\|\} \]
for all $x, y \in X$. Then $(X, \|\cdot\|)$ is called a non-Archimedean normed space (briefly, \textit{NAN-space}).

\textbf{Definition 1.2} Let $\{x_n\}$ be a sequence in a non-Archimedean normed space $X$.

(1) The sequence $\{x_n\}$ is called a Cauchy sequence if, for any $\varepsilon > 0$, there is a positive integer $N$ such that $\|x_n - x_m\| \leq \varepsilon$ for all $n, m \geq N$.

(2) The sequence $\{x_n\}$ is said to be convergent if, for any $\varepsilon > 0$, there are a positive integer $N$ and $x \in X$ such that $\|x_n - x\| \leq \varepsilon$ for all $n \geq N$. Then the point $x \in X$ is called the limit of the sequence $\{x_n\}$, which is denote by $\lim_{n \to \infty} x_n = x$.

(3) If every Cauchy sequence in $X$ converges, then the non-Archimedean normed space $X$ is called a non-Archimedean Banach space.

Note that
\[ \|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n - 1\} \]
for all $m, n \geq 1$ with $n > m$.

In this paper, using direct method we prove Hyers-Ulam-Rassias stability of the following functional equation:
\[ 3f(x + 3y) + f(3x - y) = 15f(x + y) + 15f(x - y) + 80f(y) \quad (1) \]
in non-Archimedean normed spaces.
2 Non-Archimedean Stability of the Functional Equation (1)

In this section, we prove Hyers-Ulam-Rassias stability of the functional equation (1) in non-Archimedean space. Throughout this section, let $G$ be an additive semigroup and that $X$ be a non-Archimedean Banach space.

**Theorem 2.1** Let $\zeta : G \times G \to [0, \infty)$ be a function such that
\[
\lim_{n \to \infty} |27^n \zeta \left( \frac{x}{3^n}, \frac{y}{3^n} \right) | = 0
\]
for all $x, y \in G$. Suppose that, for any $x \in G$, the limit
\[
\Phi(x) = \lim_{n \to \infty} \max \left\{ |27^{k+1} \zeta \left( \frac{x}{3^{k+1}}, 0 \right) | : 0 \leq k < n \right\}
\]
exists and $f : G \to X$ is a mapping with $f(0) = 0$ and satisfying
\[
\|3f(x + 3y) + f(3x - y) - 15f(x + y) - 15f(x - y) - 80f(y)\| \leq \zeta(x, y).
\]
Then, for all $x \in G$, $T(x) := \lim_{n \to \infty} 27^n f \left( \frac{x}{3^n} \right)$ exists and satisfying the inequality
\[
\|f(x) - T(x)\| \leq \frac{1}{|27|} \Phi(x).
\]
Moreover, if
\[
\lim_{j \to \infty} \lim_{n \to \infty} \max \left\{ |27^{k+1} \zeta \left( \frac{x}{3^{k+1}}, 0 \right) | : j \leq k < n + j \right\} = 0,
\]
then $T$ is the unique additive mapping satisfying (5).

**Proof:** Putting $y = 0$ in (4), we get
\[
\left\| 27f \left( \frac{x}{3} \right) - f(x) \right\| \leq \zeta \left( \frac{x}{3}, 0 \right)
\]
for all $x \in G$. Replacing $x$ by $\frac{x}{3^n}$ in (7), we obtain
\[
\left\| 27^{n+1} f \left( \frac{x}{3^{n+1}} \right) - 27^n f \left( \frac{x}{3^n} \right) \right\| \leq |27^n \zeta \left( \frac{x}{3^{n+1}}, 0 \right) |
\]
Thus, it follows from (2) and (8) that the sequence $\left\{ 27^n f \left( \frac{x}{3^n} \right) \right\}_{n \geq 1}$ is a Cauchy sequence. Since $X$ is complete, it follows that $\left\{ 27^n f \left( \frac{x}{3^n} \right) \right\}_{n \geq 1}$ is convergent. Set $T(x) := \lim_{n \to \infty} 27^n f \left( \frac{x}{3^n} \right)$. By induction, we can show that
\[
\left\| 27^n f \left( \frac{x}{3^n} \right) - f(x) \right\| \leq \frac{\max \left\{ |27^{k+1} \zeta \left( \frac{x}{3^{k+1}}, 0 \right) | : 0 \leq k < n \right\}}{|27|}
\]
for all \( n \geq 1 \) and \( x \in G \). By taking \( n \to \infty \) in (9) and using (3), we obtain (5). By (2) and (4), we get
\[
\left\| 3T(x + 3y) + T(3x - y) - 15T(x + y) - 15T(x - y) - 80T(y) \right\|
= \lim_{n \to \infty} \left\| 3 \cdot 27^n f \left( \frac{x + 3y}{3^n} \right) + 27^n f \left( \frac{x - 3y}{3^n} \right) - 15 \cdot 27^n f \left( \frac{x + y}{3^n} \right) \right\|
= 0
\]
for all \( x, y \in G \). Therefore, the mapping \( T : G \to X \) satisfies (1).
To prove the uniqueness property of \( T \), let \( S \) be another mapping satisfying (5). Then we have
\[
\left\| T(x) - S(x) \right\| = \lim_{j \to \infty} |27|^j \left\| T \left( \frac{x}{3^j} \right) - S \left( \frac{x}{3^j} \right) \right\|
\leq \lim_{j \to \infty} |27|^j \max \left\{ \left\| T \left( \frac{x}{3^j} \right) - f \left( \frac{x}{3^j} \right) \right\|, \left\| f \left( \frac{x}{3^j} \right) - S \left( \frac{x}{3^j} \right) \right\| \right\}
\leq \frac{1}{27} \lim_{j \to \infty} \lim_{n \to \infty} \max \left\{ |27|^{k+1} \xi \left( \frac{x}{3^{k+1}}, 0 \right) : j \leq k < n + j \right\}
= 0
\]
for all \( x \in G \). Therefore, \( T = S \). This completes the proof.

**Corollary 2.2** Let \( \xi : [0, \infty) \to [0, \infty) \) be a function satisfying
\[
\xi \left( \frac{t}{|3|} \right) < \frac{\xi(t)}{|27|}, \quad \xi(0) = 0
\]
for all \( t \geq 0 \). Let \( \kappa > 0 \) and \( f : G \to X \) be a mapping with \( f(0) = 0 \) such that
\[
\left\| 3f(x + 3y) + f(3x - y) - 15f(x + y) - 15f(x - y) - 80f(y) \right\| \leq \kappa (\xi(|x|) + \xi(|y|))
\]
for all \( x, y \in G \). Then there exists a unique cubic mapping \( T : G \to X \) such that
\[
\left\| f(x) - T(x) \right\| \leq \frac{\kappa \xi(|x|)}{|27|}.
\]
**Proof:** If we define \( \zeta : G \times G \to [0, \infty) \) by \( \zeta(x, y) := \kappa (\xi(|x|) + \xi(|y|)) \), then we have
\[
\lim_{n \to \infty} |27|^n \xi \left( \frac{x}{3^n}, \frac{y}{3^n} \right) \leq \lim_{n \to \infty} \left( |27| \xi \left( \frac{1}{|3|} \right) \right)^n \zeta(x, y) = 0
\]
Stability of additive functional equation in NAN-spaces

for all \(x, y \in G\). The last equality comes from the fact that \(|27|\xi \left( \frac{1}{|3|} \right) < 1\). On the other hand, it follows that, for all \(x \in G\),

\[
\Phi(x) = \lim_{n \to \infty} \max \left\{ |27|^{k+1} \zeta \left( \frac{x}{3^{k+1}}, 0 \right) : 0 \leq k < n \right\} = |27| \zeta \left( \frac{x}{3}, 0 \right) = \kappa \xi(|x|)
\]

exists. Also, we have

\[
\lim_{j \to \infty} \lim_{n \to \infty} \max \left\{ |27|^{j+1} \zeta \left( \frac{x}{3^{j+1}}, 0 \right) : j \leq k < n + j \right\} = \lim_{j \to \infty} |27|^{j+1} \zeta \left( \frac{x}{3^{j+1}}, 0 \right) = 0.
\]

Thus, applying Theorem 2.1, we have the conclusion. This completes the proof.

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References


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