On a New Numerical Computation of the Steady State Solution for two Infinite Server parallel Queues

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Abstract
We consider a network of two M/M/$\infty$ parallel queues having the same poissonian arrival stream with rate $\lambda$. Upon his arrival to the system a customer heads to the shortest queue and stays until being served. If the two queues have the same length, an arriving customer choose one of the two queues with the same probability. Each duration of service in the two queues is an exponential random variable with rate $\mu$ and no jockeying is permitted between the two queues. We use a linear program for a numerical computation of the steady state solution of the system.

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1 Introduction
In this paper we consider two identical M/M/$\infty$ queues in parallel with a Joining the Shortest Queue (JSQ) policy. The customers arrive to the system in accordance with a Poisson process of rate $\lambda$. Upon his arrival to the system, the customer joins the shortest queue and stays in the queue until being served. If the two queues have the same length, the customer joins one of the two queues
with the same probability. The duration of each service is an exponential random variable with rate $\mu$. No jockeying is permitted between the two queues. The goal of the present paper is the numerical computation of the steady state probability $p(i, j)$ where $i$ is the number of customers in queue 1 and $j$ the number of customers in queue 2. The shortest queue problem was initially proposed by Haight ([4]). A successful model for the symmetric case was given when the jockeying between the two queues is allowed. Kingman ([6]) obtained some asymptotic results for the joint steady state distribution of the number of customers in the two queues. Flatto and Mckean ([2]) used generating function techniques to give some limiting properties for the steady state probability. A numerical approach, using matrix geometric ([7]) technique was developed by Gertsbakh ([3]). Halfin ([5]) performed a linear programming method for the problem by giving a lower and an upper bound for the probability distribution of the total number of customers in the system. Zhao and Grassmann ([11]) used the Flatto and McKean ([2]) results to develop a numerical solution. Adan, Wessels and Zijm ([1]) showed that the steady state distribution of the queue length is a mixture of product form distributions. In order to make the problem easier, Wang and Locker ([9]) presented a model where the state space of the related Markov process was truncated into banded arrays, so they derive the probability of queue length and the customer sojourn time. Yao and Knessl ([10]) considered the model of JSQ problem with two $\text{M}/\text{M}/\infty$ queues; they perform an analytical and a numerical computation of the joint steady state solution. The numerical computation is based on solving the system of balance equations in a truncated state space involving than a truncature error for the computed probabilities. This last model is considered in the present paper with an easiest and accurate method of computation of the steady state probabilities. This kind of queues is in use in code division multiple access (CDMA) communications and in transmission systems with multiple channels. The method described here is more different than the usual numerical methods involving a truncature of the state space, so we don’t need a normalisation equation. It’s based on the fact that the steady state probabilities are expressed in terms of the diagonal probabilities. Those last probabilities are then computed with high accuracy( we further test the useful check of the accuracy as cited in section 2 of ([10]) by using a usual linear programming technique. We adapt the method called ”Méthode des convexes” ([8]) and use a linear program to perform an algorithm for the numerical computation of the joint steady state distribution. The software available for the simplex method allows us to make a computation with a desired precision for a large range of values of $\rho$. 
2 Steady state analysis of the system

Let the two M/M/∞ queues described above in the introduction. $X_t$ and $Y_t$ are the number of customers in the queue 1 and 2 respectively at time $t$. The process $(X_t, Y_t)_t$ is a recurrent positive Markov process. We note $Q = (q(e, e'))_{(e, e') \in \mathbb{N}^2 \times \mathbb{N}^2}$ the associated infinitesimal generator matrix. Let $p(i, j) = \lim_{t \to \infty} P(X_t = i, Y_t = j)$ the steady state solution for the above process. It is well known that the process $(Z_t)_t$ defined by $Z_t = X_t + Y_t$ has an M/M/∞ steady state distribution with intensity traffic $\rho = \frac{\lambda}{\mu}$, so $\lim_{t \to \infty} P(Z_t = i) = \exp(-\rho) \rho^i$. We show further that $p(i, j)$ for $i + j = n$ is a function of the diagonal probabilities $p(i, i)$ for $i \leq \lfloor \frac{n}{2} \rfloor$ where $\lfloor x \rfloor$ is the entire part of $x$. So, the computation of the steady state probabilities $p(i, j)$ for all $(i, j)$ is reduced to the computation of the diagonal probabilities $p(i, i)$.

Notations

For $k \geq 1$ we define the following $(k + 1) \times 1$ vectors and $(k + 1) \times (k + 1)$ matrices

$$
X_{2k} = \begin{pmatrix}
  p(2k, 0) \\
p(2k - 1, 1) \\
  \vdots \\
p(k + 1, k - 1) \\
p(k, k)
\end{pmatrix}, \quad X_{2k+1} = \begin{pmatrix}
  p(2k + 1, 0) \\
p(2k, 1) \\
  \vdots \\
p(k + 2, k - 1) \\
p(k + 1, k)
\end{pmatrix},
$$

$$
A_{2k} = \begin{pmatrix}
  2k & 1 & 0 & \ldots & 0 \\
  0 & 2k - 1 & 2 & \ddots & \vdots \\
  0 & 0 & \ddots & \ddots & 0 \\
  \vdots & \ddots & \ddots & k \\
  0 & \cdots & 0 & 1
\end{pmatrix}, \quad A_{2k+1} = \begin{pmatrix}
  2k + 1 & 1 & 0 & \ldots & 0 \\
  0 & 2k & 2 & \ddots & \vdots \\
  0 & 0 & \ddots & \ddots & 0 \\
  \vdots & \ddots & \ddots & k \\
  0 & \cdots & 0 & k + 1
\end{pmatrix},
$$

$$
B_2 = \begin{pmatrix}
(1 + \rho) p(1, 0) - \frac{\rho}{2} p(0, 0) \\
p(1, 1)
\end{pmatrix}, \quad B_3 = \begin{pmatrix}
(2 + \rho) p(2, 0) \\
(2 + \rho) p(1, 1) - 2\rho p(1, 0)
\end{pmatrix},
$$

and for $k \geq 2$,

$$
B_{2k} = \begin{pmatrix}
(\rho + (2k - 1)) p(2k - 1, 0) \\
(\rho + (2k - 1)) p(2k - 2, 1) - \rho p(2k - 2, 0) \\
\vdots \\
(\rho + (2k - 1)) p(k, k - 1) - \rho p(k, k - 2) - \frac{\rho}{2} p(k - 1, k - 1) \\
p(k, k)
\end{pmatrix},
$$
\[ B_{2k+1} = \begin{pmatrix} (2k + \rho) p(2k, 0) \\ (2k + \rho) p(2k - 1, 1) - \rho p(2k - 1, 0) \\ (2k + \rho) p(2k - 2, 2) - \rho p(2k - 2, 1) \\ \vdots \\ (2k + \rho) p(k + 1, k - 1) - \rho p(k + 1, k - 2) \\ (2k + \rho) p(k, k) - 2\rho p(k, k - 1) \end{pmatrix} \]

**Theorem 2.1** The steady state system of balance equations can be written as follows

\[ A_{2k} X_2 = B_2 \quad \text{and} \quad A_{2k+1} X_{2k+1} = B_{2k+1}, \quad \text{for} \ k \geq 1. \]

**Proof.** The two first balance equations give

\[
\begin{align*}
p(0, 1) &= p(1, 0) = \frac{\lambda}{2\mu} p(0, 0) = \frac{\rho}{2} p(0, 0) \\
(1 + \rho) p(1, 0) &= \frac{\rho}{2} p(0, 0) + 2p(2, 0) + p(1, 1),
\end{align*}
\]

which can be written as

\[ A_2 X_2 = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p(2, 0) \\ p(1, 1) \end{pmatrix} = \begin{pmatrix} (1 + \rho) p(1, 0) - \frac{\rho}{2} p(0, 0) \\ p(1, 1) \end{pmatrix}, \]

where \( p(0, 0) = \exp(-\rho) \).

For all \( n \geq 1 \), as showed in the transition diagram in Figure 1; the system of balance equations expresses the probabilities \( \{p(i, j) : i + j = n + 1\} \) in terms of the probabilities \( \{p(i, j) : i + j = n\} \) and \( \{p(i, j) : i + j = n - 1\} \). So we distinguish two cases:

**Case 1: \( n = 2k \) and \( k \geq 2 \).**

Because of the symmetry, the system of the \( (k + 1) \) balance equations is reduced to

\[
(\lambda + (2k - 1) \mu) p(2k - 1, 0) = (2k) \mu p(2k, 0) + \mu p(2k - 1, 1),
\]

for \( 0 < i < k - 1 \),

\[
(\lambda + (2k - 1) \mu) p(2k - 1 - i, i) = \lambda p(2k - 1 - i, i - 1) + (2k - i) \mu p(2k - i, i)
+ (i + 1) \mu p(2k - 1 - i, i + 1),
\]

and

\[
(\lambda + (2k - 1) \mu) p(k, k - 1) = \frac{\lambda}{2} p(k - 1, k - 1) + \lambda p(k, k - 2) + k \mu p(k, k)
+ (k + 1) \mu p(k + 1, k - 1).
\]
This system is equivalent to

\[(2k) p(2k, 0) + p(2k - 1, 1) = (\rho + (2k - 1)) p(2k - 1, 0),\]

for \(0 < i < k - 1,\)

\[(2k - i) p(2k - i, i) + (i + 1) p(2k - 1 - i, i + 1) = (\rho + (2k - 1)) p(2k - 1 - i, i) - \rho p(2k - 1 - i, i - 1),\]

and

\[k p(k, k) + (k + 1) p(k + 1, k - 1) = (\rho + (2k - 1)) p(k, k - 1) - \frac{\rho}{2} p(k - 1, k - 1) - \rho p(k, k - 2),\]

which is the form: \(A_{2k} X_{2k} = B_{2k}.\)

**Case 2:** \(n = 2k + 1\) and \(k \geq 1.\)

Again, because of the symmetry, the system of balance equations can be written as

\[(\lambda + 2k\mu) p(2k, 0) = (2k + 1) \mu p(2k + 1, 0) + \mu p(2k, 1),\]
for $0 < i < k$

$$(\lambda + 2k\mu)p(k + i, k - i) = \lambda p(k + i, k - i - 1)$$

$$+ (k + i + 1)\mu p(k + i + 1, k - i)$$

$$+ (k - i + 1)\mu p(k + i, k - i + 1),$$

and

$$(\lambda + 2k\mu)p(k, k) = 2\lambda p(k, k - 1) + 2(k + 1)\mu p(k + 1, k).$$

As we did for the case $n = 2k$, we isolate on the left side the terms $\{p(i, j) : i + j = 2k + 1\}$ and this leads to the form $A_{2k+1}X_{2k+1} = B_{2k+1}$. ■

**Proposition 2.2** The components of the vectors $X_{2k}$ are of the form

$$\sum_{i=1}^{k} \alpha^{(2k)}_{i,j}p(i, i) + \alpha^{(2k)}_{0,j}, 1 \leq j \leq k + 1; \alpha^{(2k)}_{i,j} \in \mathbb{R},$$

(1)

and those of $X_{2k+1}$ are of the form

$$\sum_{i=1}^{k} \alpha^{(2k+1)}_{i,j}p(i, i) + \alpha^{(2k+1)}_{0,j}, 1 \leq j \leq k + 1; \alpha^{(2k+1)}_{i,j} \in \mathbb{R},$$

(2)

**Proof.** We use a recurrence proof, the proposition is true for $k = 1$. The system of balance equations leads to

$$A_2X_2 = B_2,$$

where

$$A_2 = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, X_2 = \begin{pmatrix} p(2, 0) \\ p(1, 1) \end{pmatrix}, \text{ and } B_2 = \begin{pmatrix} (1 + \rho) p(1, 0) - \frac{\rho}{2} p(0, 0) \\ p(1, 1) \end{pmatrix}.$$

Then

$$X_2 = \begin{pmatrix} p(2, 0) \\ p(1, 1) \end{pmatrix} = A_2^{-1}B_2 = \begin{pmatrix} \frac{1}{4}\rho^2e^{-\rho} - \frac{1}{2}p(1, 1) \\ p(1, 1) \end{pmatrix}.$$

We then get $\alpha^{(2)}_{1,1} = -\frac{1}{2}, \alpha^{(2)}_{0,1} = \frac{1}{4}\rho^2e^{-\rho}$, $\alpha^{(2)}_{1,2} = 1$ and $\alpha^{(2)}_{0,2} = 0$. We also have

$$A_3 = \begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} -p(1, 1) - \frac{1}{2}\rho pp(1, 1) + \frac{1}{2}\rho^2e^{-\rho} + \frac{1}{4}\rho^3e^{-\rho} \\ 2p(1, 1) + \rho pp(1, 1) - \rho^2e^{-\rho} \end{pmatrix}.$$
So linear combinations of the components of the vector to (\(2n\) components). The same proof is valid if we assume that the proposition is true up to (2\(n\)).

\[\alpha_{1,1}^{(3)} = -\left(\frac{1}{2} + \frac{1}{4}\rho\right), \quad \alpha_{0,1}^{(3)} = \frac{1}{12} \rho^3 e^{-\rho} + \frac{1}{4} \rho^2 e^{-\rho}, \quad \alpha_{1,2}^{(3)} = \left(\frac{1}{2} + \frac{1}{4}\rho\right) \text{ and } \alpha_{0,2}^{(3)} = -\frac{1}{4} \rho^2 e^{-\rho}.\]

Then if we assume that the proposition is true up to (2\(k\) - 1) the relation \(A_{2k}X_{2k} = B_{2k}\) or \(X_{2k} = A_{2k}^{-1}B_{2k}\) gives the components of the vector \(X_{2k}\) as linear combinations of the components of the vector \(B_{2k}\) which are themselves of the form (1) or (2) (\(B_{2k}\) is given in terms of the \(X_{2k-2}\) and \(X_{2k-1}\) components). The same proof is valid if we assume that the proposition is true up to (2\(k\)).

This proposition shows also that the coefficients \(\alpha_{i,j}^{(n)}\) are obtained recursively from \(\alpha_{i,j}^{(n-1)}\) and \(\alpha_{i,j}^{(n-2)}\) for \(n \geq 4\).

**Notations**

We introduce in the following sections the real numbers denoted \(r(i, j)\) by setting

\[r(0, 0) = p(0, 0) = \exp(-\rho) \text{ and } r(0, 1) = r(1, 0) = p(0, 1) = \frac{\rho}{2} \exp(-\rho).\]

For \(i + j \geq 2\), we put \(r(i, i) = x_i \in ]0, 1[\) and \(r(i, j)\) is defined in the same way than \(p(i, j)\) so we introduce the related notations

\[Y_{2k} = \begin{pmatrix} r(2k, 0) \\ r(2k - 1, 1) \\ \vdots \\ r(k + 1, k - 1) \\ r(k, k) \end{pmatrix}, \quad Y_{2k+1} = \begin{pmatrix} r(2k + 1, 0) \\ r(2k + 1, 1) \\ \vdots \\ r(k + 2, k - 1) \\ r(k + 1, k) \end{pmatrix},\]

\[D_2 = \begin{pmatrix} (1 + \rho) r(1, 0) - \frac{\rho}{2} r(0, 0) \\ r(1, 1) \end{pmatrix}, \quad D_3 = \begin{pmatrix} (2 + \rho) r(2, 0) \\ (2 + \rho) r(1, 1) - 2\rho r(1, 0) \end{pmatrix},\]

\[D_{2k} = \begin{pmatrix} (\rho + (2k - 1)) r(2k - 1, 0) \\ (\rho + (2k - 1)) r(2k - 2, 1) - \rho r(2k - 2, 0) \\ \vdots \\ (\rho + (2k - 1)) r(k, k - 1) - \rho r(k, k - 2) - \frac{\rho}{2} r(k - 1, k - 1) \\ r(k, k) \end{pmatrix},\]

\[D_{2k+1} = \begin{pmatrix} (2k + \rho) r(2k, 0) \\ (2k + \rho) r(2k - 1, 1) - \rho r(2k - 1, 0) \\ (2k + \rho) r(2k - 2, 2) - \rho r(2k - 2, 1) \\ \vdots \\ (2k + \rho) r(k + 1, k - 1) - \rho r(k + 1, k - 2) \\ (2k + \rho) r(k, k) - 2\rho r(k, k - 1) \end{pmatrix} \]
The vectors $Y_{2k}$ and $Y_{2k+1}$ are then defined by the recursive formulas

$$A_{2k}Y_{2k} = D_{2k} \quad \text{and} \quad A_{2k+1}Y_{2k+1} = D_{2k+1}, \quad \text{for} \ k \geq 1.$$ 

So, the components of the vectors $Y_{2k}$ and $Y_{2k+1}$ are respectively of the form

$$\sum_{i=1}^{k} \alpha_{i,j}^{(2k)} r(i, i) + \alpha_{0,j}^{(2k)} \left( \text{resp.} \sum_{i=1}^{k} \alpha_{i,j}^{(2k+1)} r(i, i) + \alpha_{0,j}^{(2k+1)} \right),$$

or

$$\sum_{i=1}^{k} \alpha_{i,j}^{(2k)} x_i + \alpha_{0,j}^{(2k)} \left( \text{resp.} \sum_{i=1}^{k} \alpha_{i,j}^{(2k+1)} x_i + \alpha_{0,j}^{(2k+1)} \right),$$

where $\left( \alpha_{i,j}^{(2k)}, \alpha_{i,j}^{(2k+1)} \right) \in \mathbb{R}^2$, for $1 \leq j \leq k + 1$ and $1 \leq i \leq k$. ■

**Remark 2.3** In the next of paper we will show how the system of builted probabilities $\{r(i,j), i+j \geq 2\}$ is an accurate approximation for the system of steady state probabilities $\{p(i,j), i+j \geq 2\}$

**Proposition 2.4** For $n \geq 2$, let the $\mathbb{R}^{[n]}$ sub set $C_n$ defined as follows

$$C_n = \left\{ \left( x_1, x_2, \ldots, x_{\left[ \frac{n}{2} \right]} \right) : x_l \in ]0, 1[ , \ for \ l = 1, 2, \ldots, \left[ \frac{n}{2} \right] , \right. $$

and $r(i,j) > 0$, if $i+j = n$ and $i \neq j$,

then for $X^{(n)} = \left( x_1, x_2, \ldots, x_{\left[ \frac{n}{2} \right]} \right) \in C_n$; the set of real numbers $\{r(i,j); i+j \leq n\}$ obtained in terms of components of $X^{(n)}$ is a positive solution for the system of equilibrium equations

$$p(e) \sum_{e' \neq e} q(e, e') = \sum_{e' \neq e} p(e')q(e', e),$$

where $(e) = (i,j)$, $i+j \leq n-1$.

**Proof.** $C_n$ is nonempty while it contains the vector $(p(1,1), \ldots, p(\left[ \frac{n}{2} \right], \left[ \frac{n}{2} \right]))$, and if $x_i = p(i,i)$, for $0 \leq i \leq \left[ \frac{n}{2} \right]$, the corresponding $r(i,j)$, $i+j = n$, are exactly $p(i,j)$, $i+j = n$.

The condition $r(i,j) > 0$, if $i+j = n$, gives for those real numbers the properties of the steady state probabilities $p(i,j); i+j = n$. Then $\{r(i,j); i+j \leq n-1\}$ (from which we build up $\{r(i,j); i+j = n\}$) over the system of equilibrium equations

$$r(e) \sum_{e' \neq e} q(e, e') = \sum_{e' \neq e} r(e')q(e', e),$$
On a new numerical computation of the steady state

where \((e) = (i, j), i + j \leq n - 1\), with \((x_i)_{i \geq 0}\) playing the role of \((p(i, i))_{i \geq 0}\) have the same properties than \\{\(p(i, j); i + j \leq n - 1\)\}. So, if \(\left(x_1, x_2, \ldots, x_{\frac{n^2}{2}}\right) \in C_n\) the builded system of real numbers \\{\(r(i, j); i + j \leq n\)\} is a positive solution for the equilibrium equations

\[
    r(e) \sum_{e' \neq e} q(e, e') = \sum_{e' \neq e} r(e') q(e', e),
\]

where \((e) = (i, j), i + j \leq n - 1\).

**Corollary 2.5** If we note

\[
    S_n = \{(x_1, x_2, \ldots): x_l \in [0, 1[, l \geq 1, \text{ and } r(i, j) > 0, \text{ for } i + j = n\},
\]

then \((S_n)_n\) is a decreasing sequence of sets \(S_n \subset S_{n-1}\) and the limit \(\cap S_n\) is so that \((x_l)_{l \geq 1}\) is exactly the entire diagonal probabilities \(p(i, i), i \geq 1\).

**Proof.** Let \(\left(x_1, x_2, \ldots, x_{\frac{n^2}{2}}\right) \in S_n\). We note first that the components \(x_l\) for \(l > \left[\frac{n^2}{2}\right]\) are free from the constraints \(r(i, j) > 0\) for \(i + j = n\), then those components are identic for the two sets \(S_n\) and \(S_{n-1}\). The components \(\left(x_1, x_2, \ldots, x_{\frac{n^2}{2}}\right)\) have also to fulfill the constraints \\{\(r(i, j) > 0, i + j = n - 1\)\} (precedent proposition) then \(S_n \subset S_{n-1}\). So, when \(n\) goes to the infinity and due to the unicity of the positive solution for the infinite linear system of balance equations (the related Markov process is ergodic), the set \(S_n\) has a limit the single point of \(\mathbb{R}^N\) which is the diagonal probabilities \((p(i, i))_{i \geq 1}\).

**Remark 2.6** In order to illustrate geometrically how \((p(i, i))_{i \geq 1}\) is obtained as a decreasing sequence of the sub sets \((S_n)_n\), we sketch on a plan the behavior of the convex sets \(C_4, C_5\), and the projection of \(C_6\) on it’s two first components for \(\rho = 4\) (see Figure 2).

**Remark 2.7** In practice we need a finite number of the probabilities \(p(i, i)\). The problem is then reduced to evaluate an integer \(M\) large enough so that the sum \(\sum_{\{(i, j) / i + j \leq M\}} p(i, j)\) is close to 1. We use the fact that

\[
    \pi_n = \sum_{\{(i, j) / i + j = n\}} p(i, j) = \exp(-\rho) \frac{\rho^n}{n!}.
\]

So, for a given precision \(\epsilon\), the computation of the integer \(M\) satisfying

\[
    \left(1 - \sum_{n=0}^{M} \pi_n\right) < \epsilon,
\]
can be done easily. We note however that the system of computed probabilities \( \{r(i,j); i + j \leq M\} \) satisfies

\[
\sum_{n=0}^{M} \pi'_n = \sum_{n=0}^{M} \pi_n, 
\]

where

\[
\pi'_n = \sum_{\{i,j\}/i+j=n} r(i,j) = \exp(-\rho) \frac{\rho^n}{n!}. 
\]

Then for every \( \epsilon > 0 \); the computed probabilities \( \{r(i,j); i + j \leq M\} \) have the same properties than the steady state probabilities \( \{p(i,j); i + j \leq M\} \), so this is an improvement of the computation accuracy when comparing with the methods where a normalisation equation is needed. The determination of \( M \) is then regarded as the stopping rule for the computation algorithm described below and not a level of state space truncature which needs a normalisation equation. As an example for \( \rho = 4 \) and \( \epsilon = 10^{-10} \) we find \( M = 22 \). We then compute \( p(i,i) \) for \( 1 \leq i \leq 11 \). While the sequence \( (p(i,i))_i \) is of some interest we
sketch it in Figure 3 for some values of $\rho$ and we see that $p(i, i)$ reaches its maximum when $i$ is near $\frac{\rho}{2}$.

![Figure 3: Evolution of the diagonal probabilities](image)

3 Computation of the steady state probabilities and numerical results

3.1 Computation of $p(i, i) \ 1 \leq i \leq \left\lfloor \frac{M}{2} \right\rfloor$

While the $\left\lfloor \frac{M}{2} \right\rfloor + 1$ components of the vector $Y_M$ are of the form

$$
\sum_{i=1}^{\left\lfloor \frac{M}{2} \right\rfloor} \alpha^{(M)}_{i,j} x_i + \alpha^{(M)}_{0,j} , \ 1 \leq j \leq \left\lfloor \frac{M}{2} \right\rfloor + 1,$$

then the diagonal probabilities $p(i, i)$ are in the set solution of the system of linear inequalities:
\[
\sum_{i=1}^{[M/2]} \alpha_{i,j}^{(M)} x_i + \alpha_{0,j}^{(M)} > 0, \text{ for } 1 \leq j \leq \left\lceil \frac{M}{2} \right\rceil + 1, \text{ and } x_i > 0, \text{ for } 1 \leq i \leq \left\lceil \frac{M}{2} \right\rceil .
\]
(3)

We use the simplex algorithm to obtain a lower and an upper bound for each \( x_i \). Because of the unicity due to the ergodicity of the related Markov process, those bounds are almost equal for a large value of \( M \), we then get an accurate approximation for \( p(i, i) \) as to be seen in further computation. While \( x_i > 0 \), we can take the objective function \( \sum_{i=1}^{[M/2]} x_i \), and the constraints (3).

**Algorithm**

1. Set up a value of \( \rho \).
2. Set up a precision \( \epsilon \) and compute the first integer \( M \) satisfying
   \[
   \left( 1 - \sum_{n=0}^{M} \exp \left( -\rho \frac{\rho^n}{n!} \right) \right) < \epsilon \left( \text{or } \sum_{n=0}^{M} \exp \left( -\rho \frac{\rho^n}{n!} \right) > 1 - \epsilon \right).
   \]
3. Use the matrices \( A_i \) and vectors \( D_i \) defined in notation 3 in order to get from a recursion the vector \( Y_M \) having the components denoted by
   \[
   \sum_{i=1}^{[M/2]} \alpha_{i,j}^{(M)} x_i + \alpha_{0,j}^{(M)}, \quad 1 \leq j \leq \left\lceil \frac{M}{2} \right\rceil + 1
   \]
   (the coefficients \( \alpha_{i,j}^{(M)} \) are then derived from this recursion).
4. Use the simplex algorithm with objective function \( \sum_{i=1}^{[M/2]} x_i \) and constraints
   \[
   \sum_{i=1}^{[M/2]} \alpha_{i,j}^{(M)} x_i + \alpha_{0,j}^{(M)} \geq 0, \quad 1 \leq j \leq \left\lceil \frac{M}{2} \right\rceil + 1 \text{ and } x_i \geq 0, \quad 1 \leq i \leq \left\lceil \frac{M}{2} \right\rceil .
   \]
5. Get a lower and an upper bound for \( x_i \) denoted respectively \( \max x_i \) and \( \min x_i \) and put \( x_i = \frac{1}{2} (\max x_i + \min x_i) \) which is the approximation of \( p(i, i) \).
6. Return to the matrix formulation for the computation of \( p(i, j) \), \( i + j \leq M \).
3.2 Numerical results

We start the computation of the probabilities \( r(i, j) \) for a small value of \( \rho \). For \( \rho = 1 \) and \( \epsilon = 10^{-10} \), we get \( M = 12 \). so the computed probabilities \( r(i, j) \), for \( i + j \leq 12 \) is given in the Table 1 (for convenience of presentation, we give the results up to two decimals). As an other case of computation for a relatively large value of intensity traffic, we take \( \rho = 40 \), but for simple presentation we report only some values of the \( r(i, j) \) given in the Table 3. In this case, for a ten digits computation set up, we get \( M = 85 \), so the method allows us to compute the set of probabilities \( \{ r(i, j) \colon i + j \leq 85 \} \). We then give \( r(i, j) \) for \( N = i + j = 10, 20, 30, 40, 50 \) and 60. Further we make, in tables 3 and 4, a comparison between the theoretical probabilities 

\[ \pi_k = \frac{\rho^k}{k!} \exp(-\rho), \]

and the computed probabilities \( \sum_{i+j=k} r(i, j) \) for the two cases: \( \rho = 1 \) and \( \rho = 40 \).

**Remark 3.1** We see that the numerical and computed results for the probabilities \( \pi_N \) are equal. This is due to the high accuracy with which the diagonal probabilities are computed and the fact that the computed probabilities \( \{ r(i, j) \colon i + j = n \} \) have exactly the same properties than the steady state probabilities \( \{ p(i, j) \colon i + j = n \} \). We give below (Table 5) the bounds for some \( x_i \) given by the simplex method for \( \rho = 1 \) and \( \rho = 40 \).

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*Table 1: The \( r(i, j) \)’s for \( \rho = 1 \)
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Table 2: A sample of \(r(i,j)\)'s for \(p = 40\)
On a new numerical computation of the steady state

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<th>$k$</th>
<th>$\pi_k = \frac{e^{-1}}{k!}$</th>
<th>$\sum_{(i,j)/i+j=k} r(i,j)$</th>
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*Table 3:* Comparison between the computed $\pi'_k$ and theoretical $\pi_k$ for $\rho=1$

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<th>$\frac{e^{-40} 40^N}{N!}$</th>
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<td>6.786492198 $10^{-4}$</td>
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<tr>
<td>105</td>
<td>5.830469527 $10^{-2}$</td>
<td>5.830469527 $10^{-2}$</td>
<td>65</td>
<td>7.011160453 $10^{-5}$</td>
<td>7.011160453 $10^{-5}$</td>
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<tr>
<td>110</td>
<td>6.137336344 $10^{-2}$</td>
<td>6.137336344 $10^{-2}$</td>
<td>70</td>
<td>4.943278525 $10^{-6}$</td>
<td>4.943278525 $10^{-6}$</td>
</tr>
<tr>
<td>115</td>
<td>6.294703942 $10^{-2}$</td>
<td>6.294703942 $10^{-2}$</td>
<td>75</td>
<td>2.444040225 $10^{-7}$</td>
<td>2.444040225 $10^{-7}$</td>
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<tr>
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<td>6.294703942 $10^{-2}$</td>
<td>80</td>
<td>8.675455923 $10^{-9}$</td>
<td>8.675455923 $10^{-9}$</td>
</tr>
</tbody>
</table>

*Table 4:* Comparison between the computed $\pi'_k$ and theoretical $\pi_k$ for $\rho = 40$
Table 5: A sample of the computed bounds for $x'_i$'s.

<table>
<thead>
<tr>
<th>$\rho = 1$</th>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\min x_1$</td>
<td>0.16328443070905508503672032044000922189295</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\max x_1$</td>
<td>0.16328443070905508503672032044000922189295</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\min x_5$</td>
<td>0.97017465928403972300912856099958805441058 $10^{-1}$</td>
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<tr>
<td>$\max x_5$</td>
<td>0.97017465928403972300912856099958805445255 $10^{-1}$</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>$\min x_{12}$</td>
<td>0.58132007277538691906756103192909749773896 $10^{-24}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\max x_{12}$</td>
<td>0.5813200727753869190685686873406199264901973 $10^{-24}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\rho = 40$</th>
<th></th>
<th></th>
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</thead>
<tbody>
<tr>
<td>$\min x_1$</td>
<td>0.20110647637112425846 $10^{-14}$</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$\max x_1$</td>
<td>0.20110647637112436542 $10^{-14}$</td>
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<tr>
<td>$\min x_{15}$</td>
<td>0.01164145095269965967</td>
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</tr>
<tr>
<td>$\max x_{15}$</td>
<td>0.01164145095271687431</td>
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</tr>
<tr>
<td>$\min x_{20}$</td>
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</tr>
<tr>
<td>$\max x_{20}$</td>
<td>0.04301571448628082649</td>
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</tr>
</tbody>
</table>

Remark 3.2 From the Table 5 we see that if we note $\epsilon_{1,i} = \max x_i - \min x_i$ then $\epsilon_1 = \max_{i} \epsilon_{1,i}$ is around the precision $\epsilon$ cited above and this is valid for all the values of $\rho$. So $p(i,j)$ is computed up to $\epsilon$.

4 Conclusion

This new method of computation for the steady state solution of two infinite server parallel queues is of some interest regarding it’s simplicity and powerfulness computation. Just useful mathematics as classic linear algebra and simplex method are needed. At the step M, the formal computation of the vector $Y_M$ needs the use of sparse matrices. Also, it’s well known that the complex algorithm is an efficient algorithm for solving linear systems. Then, the computation time needed by the method can be considered as one of it’s advantages. It can also be adapted for a large class of queuing models specially those with the JSQ policy; this is due to the same linear structure of the system of balance equations of those models. At the first time for example, we can add for each queue an own arrival stream and the method remains in use. So we can do for the general shortest queue problem with its complicated situations as the nonsymmetric case where each queue has also it’s own arrival rate in addition to the common one.
References


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