New Pricing Formula for Arithmetic Asian Options

Using PDE Approach

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Abstract

Pricing and hedging Arithmetic Asian options are difficult, since at present there is no closed-form analytical solution exists to price them. This difficulty has led to the development of various methods and models used to price these instruments. In this paper, we propose partial differential equations (PDEs) approach to value the continuous arithmetic Asian options. We provide analytical solution for Rogers and Shi PDE [9] by using changes of some variables and Fourier transform. This is the contribution of this paper since to the best of our knowledge; Rogers-Shi PDE has only been solved numerically. We transform the second order PDE of the arithmetic Asian options to the ODE from the first order, and then we give it’s analytical solution.

1. Introduction

Asian options are path-dependent option whose payoff depends on the average value of the underlying assets during a specific set of dates across the life of the option. Because the payoff of the Asian options depends on the average value of the underlying asset, volatility in the average value tends to be smoother and lower than that of the plain vanilla options. Therefore, Asian options, which tend to be less expensive than comparable Plain vanilla put or calls appear very attractive. The two basic forms of
averages in Asian options (arithmetic and geometric), both can be structured as calls or puts. A geometric average Asian option is easy to price because a closed-form solution is available [1]. However the most difficult one is arithmetic type, which is the most commonly used, though an exact analytical solution for arithmetic average rate Asian options does not exist. This missing solution is primarily because the arithmetic average of a set of lognormal random variables is not lognormally distributed.

Since no general analytical solution for the price of the arithmetic Asian option is known, a variety of techniques have been developed to analyze arithmetic average Asian options and approach the problem of its valuation some of them:

1. Monte Carlo simulations.
2. Binomial trees and lattices.
3. The PDE approach.
4. General numerical methods.
5. Analytical approximations.

This work follows the PDE approach. There are many authors have studied the PDE of Asian options, most recently, Geman and Yor [7] use Laplace transform in time of the Asian option price. However, this transform exist for some cases. Rogers and Shi [9] transform the problem of valuing Asian options to the problem of solving a parabolic equation in two variables from the second order. However it is difficult to use numerical methods to solve this PDE, and they derive lower-bound formulas for Asian options by computing the expectation based on some zero-mean Gaussian variable. Zhang [11] present a theory of continuously-sampled Asian option pricing, and he solves the PDE with perturbation approach. Vecer’s approach [10] is based on Asian option as option on a traded account; he provides a one-dimensional PDE for Asian options. Francois and Tony [6] derive accurate and fast numerical methods to solve Rogers and Shi PDE. Chen and Lyuu [2] develop the lower-bound pricing formulas of Rogers and Shi PDE [9] to include general maturities instead one year. Cruz-Baez and Gonzalez-Rodrigues [3] obtain the same solution of Geman and Yor for arithmetic Asian options using Partial differential equations, integral transforms, and Mathematica programming, instead Bessel processes. Dewynne and Shaw [4] provide a simplified means of pricing arithmetic Asian options by PDE approach, they derive an analytical formula for the Laplace transform in time of the Asian option, and they obtain asymptotic solutions for Black-Scholes PDE for Asian options for low-volatility limit which is the big problem on using Laplace transform. Elshegmani and Ahmad [5] provide a new method for solving arithmetic Asian option PDE using Mellin transform; they reduce the second order PDE to the first order and the give its final solution.

In this work we consider the PDE approach, and give a simple solution for the PDE of Asian option which proposed by Rogers and Shi, using Fourier transform.
2. Transformation techniques

The value of the arithmetic Asian options is characterized by the following PDE with a boundary condition [6]:

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} + \frac{1}{t} (S - A) \frac{\partial V}{\partial A} - r V = 0, \\
V(T, S, A) = \phi(S, A),
\]

(2.1)

where \(S\) is a stock price, \(r\) is a constant interest rate, \(\sigma\) is a constant asset volatility, \(T\) is the expiration date, and \(A = \frac{1}{t} \int_0^t S(t) \, dt\) is the average of the stock price at time \(t\).

There are four different types of the arithmetic Asian options according to the payoff function:

1) Fixed strike call option
   \[
   \phi(S, A) = (A - k)^+
   \]

2) Fixed strike put option
   \[
   \phi(S, A) = (k - A)^+
   \]

3) Floating strike call option
   \[
   \phi(S, A) = (S - A)^+
   \]

4) Floating strike put option
   \[
   \phi(S, A) = (A - S)^+
   \]

where \(k\) is a strike price, also named an exercise price.

For the case of fixed strike call option with exercise price \(k\), the call option at time \(t\) is

- In-the-money if \(A(t) > k\),
- At-the-money if \(A(t) = k\),
- Out-of-the-money if \(A(t) < k\),

and vice versa for fixed strike put option, the put option is

- In-the-money if \(A(t) < k\),
- At-the-money if \(A(t) = k\),
• Out-of-the-money if $A(t) > k$.

In this work we propose the following transformation, which allow us to transform the PDE (2.1) to a simpler parabolic equation in two variables

$$ V(t, S, A) = S f(t, \xi), \quad \xi = \frac{k - tA}{T}, $$

\[\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 \xi^2 \frac{\partial^2 f}{\partial \xi^2} - \left( \frac{1}{T} + r \xi \right) \frac{\partial f}{\partial \xi} = 0, \quad (2.2)\]

for the case of fixed strike call option $\phi(\xi) = \max(-\xi, 0) = \xi^-$, fixed strike put option $\phi(\xi) = \max(\xi, 0) = \xi^+$, floating strike call $\phi(\xi) = \max(\xi, 0) = (1 + \xi)^-$, and floating strike put $\phi(\xi) = \max(\xi, 0) = (1 + \xi)^+$. Note that, Eq. (2.2) has been obtained by Rogers and Shi [9] using different approaches.

From the above transformation we can see that, if $\xi = 0$ then the option is at-the-money, and if $\xi > 0$, the option is in-the-money. In this work we will price or value only the options which are in-the-money, which are the most important, as options at-the-money or out-of-the-money are worthless. Also Rogers and Shi [9] obtain the analytical solution for the case of $\xi \leq 0$,

$$ f(t, \xi) = \frac{1}{rT} \left( 1 - e^{-r(T-t)} \right) - \xi e^{-r(T-t)}. $$

So that, we shall consider only the case when $\xi > 0$. Under the substitution

$$ |x| = \ln \xi, \quad \xi > 0, $$

Eq. (2.2) becomes

\[\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial x^2} - \left( \frac{1}{2} \sigma^2 + r + \frac{e^{1/2}}{T} \right) \frac{\partial f}{\partial x} = 0, \quad (2.3)\]

$$ f(T, x) = \phi(x). $$
We can see that, most of the coefficients in the above equation are constants except one, and this coefficient is exponential function, so we can easily use Fourier transform. Fourier transform for a function \( f(x) \) is defined by

\[
F \{ f(x) \} = g(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x} \, dx,
\]

and the inverse Fourier transform is

\[
F^{-1} \{ g(\omega) \} = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\omega)e^{i\omega x} \, d\omega,
\]

\[
F \left\{ \frac{\partial^n f}{\partial x^n} \right\} = (i\omega)^n g(\omega),
\]

\[
F \{ e^{+it} \} = \int_{-\infty}^{\infty} e^{-i\omega t} \, d\omega = \int_{-\infty}^{0} e^{(1-i\omega)t} \, dx + \int_{0}^{\infty} e^{(1+i\omega)t} \, dx = \frac{2}{1+\omega^2}, \quad x > 0,
\]

and

\[
F \{ f_1(x)f_2(x) \} = g_1(\omega) * g_2(\omega),
\]

where * denotes the convolution operation.

Applying Fourier transform in \( x \) on Eq. (2.3)

\[
\frac{\partial h(t, \omega)}{\partial t} - \left( \frac{i\omega \sigma^2}{2} + i\omega + \frac{\sigma^2 \omega^2}{2} + \frac{i\omega}{T} \frac{2}{1+\omega^2} \right) h(t, \omega) = 0,
\]

\[
h(T, \omega) = \phi(\omega).
\]

Let \( \tau = T - t \) where \( T \) is the expiration date, then Eq. (2.4) is transformed to

\[
\frac{\partial h(\tau, \omega)}{\partial \tau} = -\left( \frac{i\omega \sigma^2}{2} + i\omega + \frac{\sigma^2 \omega^2}{2} + \frac{i\omega}{T} \frac{2}{1+\omega^2} \right) h(t, \omega),
\]

\[
h(0, \omega) = \phi(\omega)
\]

The above equation is an ordinary differential equation from the first order with initial condition, its solution is:
\[ h(\tau, \omega) = \varphi(\omega) \exp \left\{ -\frac{i\omega \sigma^2}{2} - \frac{\sigma^2 \omega^2}{2} - \left( \frac{i\omega}{T} \star \frac{2}{1 + \omega^2} \right) \right\} \tau. \quad (2.6) \]

Applying inverse Fourier transform on Eq. (2.6) to get the function \( f(t,x) \).

\[ f(t,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\omega) e^{ix\omega} \exp \left\{ -\frac{i\omega \sigma^2}{2} - \frac{\sigma^2 \omega^2}{2} - \left( \frac{i\omega}{T} \star \frac{2}{1 + \omega^2} \right) \right\} (T-t) d\omega. \quad (2.7) \]

We have assumed that \(|x| = \ln \xi, \xi > 0, x = \pi \ln \xi, \) this equation has a symmetric graph about \( \xi \)-axis, so that, we will take \( x = +\ln \xi, x \geq 0 \) and multiples the integration by 2.

\[ f(t,\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\omega) \xi^{i\omega} \exp \left\{ -\frac{i\omega \sigma^2}{2} - \frac{\sigma^2 \omega^2}{2} - \left( \frac{i\omega}{T} \star \frac{2}{1 + \omega^2} \right) \right\} (T-t) d\omega. \quad (2.8) \]

The solution of the arithmetic Asian option PDE then is given by

\[ V(t,S,A) = S f(t,\xi), \quad \xi = \frac{k - \frac{tA}{T}}{S}, \]

where \( f(t,\xi) \) satisfied Eq. (2.8).
To prove that, our solution (2.8) is a direct solution for Eq. (2.2), derivative Eq. (2.8) with respect to all variables, and then substitute them into Eq. (2.2)

\[
\frac{\partial f}{\partial t} = \frac{1}{\pi} \int_{-\infty}^{\infty} \phi(\omega) \xi^{\omega} \left[ \frac{i\omega \sigma^2}{2} + i\omega + \frac{\sigma^2 \omega^2}{2} + \left( \frac{i\omega}{T} \ast \frac{2}{1 + \omega^2} \right) \right] 
\exp \left\{ - \frac{i\omega \sigma^2}{2} - i\omega - \frac{\sigma^2 \omega^2}{2} - \left( \frac{i\omega}{T} \ast \frac{2}{1 + \omega^2} \right) \right\} (T-t) \, d\omega,
\]

(2.9)

\[
\frac{\partial f}{\partial \xi} = -\frac{1}{\pi} \int_{-\infty}^{\infty} \phi(\omega) i\omega \xi^{\omega-1} \exp \left\{ - \frac{i\omega \sigma^2}{2} - i\omega - \frac{\sigma^2 \omega^2}{2} - \left( \frac{i\omega}{T} \ast \frac{2}{1 + \omega^2} \right) \right\} (T-t) \, d\omega,
\]

(2.10)

\[
\frac{\partial^2 f}{\partial \xi^2} = -\frac{1}{\pi} \int_{-\infty}^{\infty} \phi(\omega) (\omega^2 + i\omega) \xi^{\omega-2} \exp \left\{ - \frac{i\omega \sigma^2}{2} - i\omega - \frac{\sigma^2 \omega^2}{2} - \left( \frac{i\omega}{T} \ast \frac{2}{1 + \omega^2} \right) \right\} (T-t) \, d\omega.
\]

(2.11)

Substituting equations (2.9) – (2.11) into Eq. (2.2) yield

\[
\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 \xi \frac{\partial^2 f}{\partial \xi^2} - \left( \frac{1}{T} + r \xi \right) \frac{\partial f}{\partial \xi} = \frac{1}{\pi} \int_{-\infty}^{\infty} \phi(\omega) \xi^{\omega} \left[ \frac{i\omega \sigma^2}{2} + i\omega + \frac{\sigma^2 \omega^2}{2} + \left( \frac{i\omega}{T} \ast \frac{2}{1 + \omega^2} \right) - \frac{\sigma^2}{2} (\omega^2 + i\omega) - ri\omega - \frac{i\omega}{T} \xi^{-1} \right] 
\exp \left\{ - \frac{i\omega \sigma^2}{2} - i\omega - \frac{\sigma^2 \omega^2}{2} - \left( \frac{i\omega}{T} \ast \frac{2}{1 + \omega^2} \right) \right\} (T-t) \, d\omega,
\]

from the Fourier transform we have

\[
F \left\{ \xi^{-1} \right\} = F \left\{ e^{\frac{-\xi}{1 + \omega^2}} \right\} = \frac{2}{1 + \omega^2}.
\]
The last equality because

\[
\frac{i\omega \sigma^2}{2} - \frac{r i\omega - \sigma^2 \omega^2}{2} + \left( \frac{i\omega}{T} \right) \frac{2}{1+\omega^2} \left( \omega + i\omega \right) - ri\omega - \left( \frac{i\omega}{T} \right) \frac{2}{1+\omega^2} = 0.
\]

So that, our solution is a direct solution of the arithmetic Asian option PDE.

**Conclusion**

We have studied partial differential equation arising in the valuation of arithmetic Asian option. We approach the problem of computing the price of all types of arithmetic Asian options. We consider the PDE of Rogers and Shi for arithmetic Asian options which has been studied by many authors in the literatures, because it is not easy to solve, most of them tried to improve its numerical solution. In this paper we have obtained a direct solution for Rogers and Shi PDE for arithmetic Asian options using means of partial differential equation and Fourier transform. We have shown that, the second order PDE of arithmetic Asian option in three variables can be transformed to ODE from the first order with an initial condition, which can be solved analytically.

**References**


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