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On the Interval Zoro Symmetric Single-step Procedure for Simultaneous Finding of Polynomial Zeros

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Abstract

The aim of this paper is to present the interval zoro symmetric single-step procedure *IZSS1* which is the modification of interval symmetric single-step procedure *ISS1*. This procedure has a faster convergence rate than does *ISS1*. We start with suitably chosen initial disjoint intervals where each interval contains a zero of a polynomial. The *IZSS1* method will produce successively smaller intervals that are guaranteed to still contain the zeros. The convergence rate of the procedure *IZSS1* will be shown in this paper. The procedure is run on five test polynomials and the results obtained show that the modified method is better in comparison with the procedure *ISS1*.

Keywords: Interval analysis; Interval procedure; Simultaneous inclusion; Simple zeros; R-order of convergence; R-factor of a sequence

1 Introduction

The iterative procedures for the simultaneous determination of all zeros of a polynomial are the important root solvers since they overcome deflation (McNamee [**?**]).These procedures start with some disjoint intervals which contain the polynomial zeros. The interval iterative procedures will determine bounded intervals each of which is guaranteed to still contain the zero. It is a very significant way of obtaining reliable bounds on the zeros as the intervals sequences generated by the procedures are always converges to the zeros. Since then, researchers have concentrated on iterative methods such as the famous Newton's method; see for examples, Gargantini and Henrici (1971), Gargantini (1978), Petkovic (1982), Alefeld and Herzberger (1983), Milovanovic and Petkovic (1983), Petkovic and Stefanovic (1986), Monsi(1988) ,Carstensen (1993), Sun and Li (1999).

In this paper, we will describe the interval zoro symmetric single-step procedure *IZSS1*. The significant of using the interval analysis for determining the convergence rate of the procedure is that the analysis is very straight forward compared to the analysis of the point procedures. We use the Intlab V5.5 toolbox (S.M. Rump [?]), for MATLAB R2007a in order to determine the numerical results. It is a necessary tool to determine the narrow computationally rigorous bounds on the zeros of the polynomials.

The R-order of convergence analysis of an iterative procedure used in this paper is as a measure of the asymptotic convergence rate of the procedure. The concept of R-order of convergence is discussed in detail in Ortega and Rheinboldt [?] and Alefeld and Herzberger [?]. The R-order of the procedure I which converge to x^* is denoted by $O_R(I, x^*)$ and the R-factor of a null sequence $w^{(k)}$ generated from the procedure I is denoted by $R_p(w^{(k)})$, where $p \geq 1$ and $w^{(k)}$ is a null sequence generated from the procedure I.

The following theorem is proved in Ortega and Rheinboldt [?].

Theorem 1.1 *Let I be an iteration procedure with the limit x^*, and let $\Omega(I, x^*)$ be the set of all sequences $(x^{(k)})$ generated by I having the properties that $\lim_{x\to\propto} x^{(k)} = x^*$ and $x^* \subseteq x^{(k)}, k \geq 0$. If there exists a $p \geq 1$ and a constant γ such that for all $\left\{x^{(k)}\right\} \epsilon\Omega(I, x^*)$ and for a norm $\|\cdot\|$, it holds that $\left\|h^{(k+1)}\right\| \geq \gamma \left\|h^{(k)}\right\|^p$, $k \geq k(\left\{x^{(k)}\right\})$then it follows that the R-order of I satisfies the inequality $O_R(I, x^*) \geq p$.* ■

Furthermore, if there exist a $p \geq 1$ such that for any null sequence $\left\{w^{(k)}\right\}$ generated from $\left\{x^{(k)}\right\}$ then the R-factor of such sequence is defined to be

$$R_p\left(w^{(k)}\right) = \begin{cases} \underset{k\to\infty}{lim}\, sup \left\|w^{(k)}\right\|^{\frac{1}{k}}, p = 1 \\ \underset{k\to\infty}{lim}\, sup \left\|w^{(k)}\right\|^{\frac{1}{p^k}}, p > 1 \end{cases},$$

where R_p is independent of the norm $\|\cdot\|$. Suppose that $R_p\left(w^{(k)}\right) < 1$ then it follows from Ortega and Rheinboldt [?] that the R-order of I satisfies the inequality $O_R(I, x^*) \geq p$. We will use this result in order to calculate the R-order of convergence of *IZSS1* in the subsequent section.

2 The Simultaneous Inclusion Procedures of Real Zeros of Polynomials.

Let $p : R^1 \to R^1$ be a polynomial of degree n defined by

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$$
$$= \sum_{i=0}^{n} a_i x^i \tag{1}$$

where $a_i \in R^1 (i = 1, ..., n)$ are given. Suppose that p has n distinct zeros $x_i^* \in R$ $(i = 1, ..., n)$ and that $X_i^{(0)} \in I(R)$ (the set of real intervals) $(i = 1, ..., n)$ are such that

$$x_i^* \in X_i^{(0)} \quad (i = 1, ..., n) \tag{2}$$

and

$$X_i^{(0)} \cap X_j^{(0)} = \phi \quad (i, j = 1, ..., n; i \neq j), \tag{3}$$

It assumed henceforth that $a_n = 1$, so that

$$p(x) = \prod_{j=1}^{n} (x - x_j^*). \tag{4}$$

By (4), for $i = 1, ..., n$ $(\forall x \neq x_j^* (j = 1, ..., n))$

$$x_i^* = x - \frac{p(x)}{\prod_{j \neq 1} (x - x_j^*)}. \tag{5}$$

If

$$x_i^{(0)} = m(X_i^{(0)}) \quad (i = 1, ..., n), \tag{6}$$

are the midpoints of the interval $X_i^{(0)}$ $(i = 1, ..., n)$ respectively. Then by (2) and (3),

$$x_i^{(0)} \neq x_j^* \quad (i = 1, ..., n; j \neq i). \tag{7}$$

So by (5) we have,

$$x_i^* = x_i^{(0)} - \frac{p(x_i^{(0)})}{\prod_{j \neq 1} (x_i^{(0)} - x_j^*)} \quad (i = 1, ..., n). \tag{8}$$

Furthermore, by (3), (6), $x_i^{(0)} \notin X_j^{(0)} \quad (i,j = 1,...,n; j \neq i)$, whence

$$0 \notin \prod_{j\neq i}(x_i^{(0)} - X_j^{(0)}) \quad (i = 1,...,n). \tag{9}$$

So, by (2), (8), and the inclusion monotonicity (Alefeld and Herzberger [?]) of real interval arithmetic,

$$x_i^* \in X_i^{(1)} = \left\{ x_i^{(0)} - \frac{p(x_i^{(0)})}{\prod_{j\neq i}(x_i^{(0)} - X_j^{(0)})} \right\} \cap X_i^{(0)} \quad (i = 1,...,n). \tag{10}$$

This gives rise to the interval total step procedure IT of Alefeld and Herzberger defined by

$$x_i^{(k)} = m(X_i^{(k)}) \quad (i = 1,...,n), \tag{11a}$$

$$X_i^{(k+1)} = \left\{ x_i^{(k)} - \frac{p(x_i^{(k)})}{\prod_{j\neq i}(x_i^{(k)} - X_j^{(k)})} \right\} \cap X_i^{(k)} \ (i = 1,...,n) \ (k \geq 0), \tag{11b}$$

The rate of convergence of IT procedure is at least two or $O_R(IT, x_i^*) \geq 2$. (Alefeld and Herzberger [?])

A modification of IT is known as single-step procedure IS (Alefeld and Herzberger [?]) and it is defined by ($for \ \ k \geq 0$)

$$x_i^{(k)} = m(X_i^{(k)}) \quad (i = 1,...,n), \tag{12a}$$

$$X_i^{(k+1)} = \left\{ x_i^{(k)} - \frac{p(x_i^{(k)})}{\prod_{j=1}^{i-1}(x_i^{(k)} - X_j^{(k+1)}) \prod_{j=i+1}^{n}(x_i^{(k)} - X_j^{(k)})} \right\} \cap X_i^{(k)} \tag{12b}$$

$$(i = 1,...,n),$$

It have been proved in Alefeld and Herzberger [?] that for $i = 1,...n$, $O_R(IS, x_i^*) \geq 1 \pm \sigma$. where $\sigma \in (1,2)$. is the greatest positive zero of $t^n - t - 1$.

The natural extension of the interval single-step procedure IS is the Interval symmetric single-step procedure *ISS1* of Monsi [?] and it is defined by ($for \ \ k \geq 0$)

$$x_i^{(k)} = m(X_i^{(k)}) \quad (i = 1,...,n), \tag{13a}$$

$$X_i^{(k,1)} = \left\{ x_i^{(k)} - \frac{p(x_i^{(k)})}{\prod_{j=1}^{i-1}(x_i^{(k)} - X_j^{(k,1)}) \prod_{j=i+1}^{n}(x_i^{(k)} - X_j^{(k,0)})} \right\} \cap X_i^{(k,0)} \tag{13b}$$

$$(i = 1,...,n),$$

$$X_i^{(k+1)} = \left\{ x_i^{(k)} - \frac{p(x_i^{(k)})}{\prod_{j=1}^{i-1}(x_i^{(k)} - X_j^{(k,1)}) \prod_{j=i+1}^{n}(x_i^{(k)} - X_j^{(k+1)})} \right\} \cap X_i^{(k,1)} \tag{13c}$$

$$(i = n,...,1),$$

The rate of convergence of procedure *ISS1* is at least three or $O_R(ISS1, x_i^*) \geq 3$ (Alefeld and Herzberger [?]).

3 The Interval Zoro Symmetric Single-step Procedure

An extension of the idea of Aitken (1950), Alefeld (1977) and Monsi (1988) are used to establish the new modified method so-called the interval zoro symmetric single-step procedure *IZSS1*. The 'zoro' is referred to the pattern of the 'z' steps in the procedure. The procedure *IZSS1* consists of generating the sequences $(X_i^{(k)})\quad(i = 1, ..., n)$ from

$$X_i^{(1,0)} = X_i^{(0)} \quad (initial\ intervals) \tag{14a}$$

$$for\ k \geq 1, \tag{14b}$$

$$x_i^{(k)} = m(X_i^{(k,0)}) \quad (i = 1, ..., n), \tag{14c}$$

$$X_i^{(k,1)} = \left\{ x_i^{(k)} - \frac{p(x_i^{(k)})}{\prod_{j=1}^{i-1}(x_i^{(k)} - X_j^{(k,1)}) \prod_{j=i+1}^{n}(x_i^{(k)} - X_j^{(k,0)})} \right\} \cap X_i^{(k,0)} \quad (i = 1, ..., n), \tag{14d}$$

$$X_i^{(k,2)} = \left\{ x_i^{(k)} - \frac{p(x_i^{(k)})}{\prod_{j=1}^{i-1}(x_i^{(k)} - X_j^{(k,1)}) \prod_{j=i+1}^{n}(x_i^{(k)} - X_j^{(k,2)})} \right\} \cap X_i^{(k,1)} \quad (i = n, ..., 1), \tag{14e}$$

$$X_i^{(k,3)} = \left\{ x_i^{(k)} - \frac{p(x_i^{(k)})}{\prod_{j=1}^{i-1}(x_i^{(k)} - X_j^{(k,3)}) \prod_{j=i+1}^{n}(x_i^{(k)} - X_j^{(k,2)})} \right\} \cap X_i^{(k,2)} \quad (i = 1, ..., n), \tag{14f}$$

$$X_i^{(k+1)} = X_i^{(k,3)} \quad (i = n, ..., 1), \tag{14g}$$

$$X_i^{(k+1,0)} = X_i^{(k+1)} \quad (i = 1, ..., n), \tag{14h}$$

Theorem 3.1 *If (i) (2) and (3) holds; (ii) the sequences* $X_i^{(k)}$ $(i = 1, ..., n)$, *are generated from (14), then* $(\forall k \geq 0)$ $x_i^* \in X_i^{(k+1)} \subseteq X_i^{(k)}$ $(i = 1, ..., n)$, *If also (iii)* $0 \notin D_i \in I(R)$ *is such that* $p'(x) \in D_i(\forall x \in X_i^{(0)})$ $(i = 1, ..., n)$, *then* $X_i^{(k)} \to x_i^*$ $(k \to \infty)$ $(i = 1, ..., n)$ *and* $w(X_i^{(k+1)}) \leq \frac{1}{2}(1 - \frac{d_{i_I}}{d_{i_S}})w(X_i^{(k)})$ *holds. Then for* $(i = 1, ..., n)$, $O_R(IZSS1, x_i^*) \geq 4$. ■

Proof The proof that $x_i^* \in X_i^{(k+1)} \subseteq X_i^{(k)}$ $(i = 1, ..., n)$ $(\forall k \geq 0)$ and that (12) holds is almost identical with the corresponding proofs in Alefeld and Herzberger [?], and is therefore omitted. It remains to prove that for

$(i = 1, ..., n)$, $O_R(IZSS1, x_i^*) \geq 4$.
From (Alefeld and Herzberger [?]) it may be shown that $\exists \alpha > 0$ such that $(\forall k \geq 0)$,

$$w_i^{(k,1)} \leq \beta w_i^{(k,0)} \left\{ \sum_{j=1}^{i-1} w_j^{(k,1)} + \sum_{j=i+1}^{n} w_j^{(k,0)} \right\} \quad (i = 1, ..., n), \tag{15}$$

$$w_i^{(k,2)} \leq \beta w_i^{(k,0)} \left\{ \sum_{j=1}^{i-1} w_j^{(k,1)} + \sum_{j=i+1}^{n} w_j^{(k,2)} \right\} \quad (i = n, ..., 1), \tag{16}$$

and

$$w_i^{(k,3)} \leq \beta w_i^{(k,0)} \left\{ \sum_{j=1}^{i-1} w_j^{(k,3)} + \sum_{j=i+1}^{n} w_j^{(k,2)} \right\} \quad (i = 1, ..., n), \tag{17}$$

where

$$w_i^{(k,s)} = (n-1)\alpha w(X_i^{(k,s)}) \quad (s = 0, 1, 2, 3), \tag{18}$$

and

$$\beta = \frac{1}{n-1}. \tag{19}$$

Let

$$u_i^{(1,1)} = \begin{cases} 2 & (i = 1, ..., n-1) \\ 3 & (i = n) \end{cases}, \tag{20}$$

$$u_i^{(1,2)} = \begin{cases} 3 & (i = n, ..., 2) \\ 4 & (i = 1) \end{cases}, \tag{21}$$

and

$$u_i^{(1,3)} = \begin{cases} 4 & (i = 1, ..., n-1) \\ 5 & (i = n) \end{cases}, \tag{22}$$

and for $(r = 1, 2, 3)$, let

$$u_i^{(k+1,r)} = \begin{cases} 4u_i^{(k,r)} & (i = 1, ..., n-1) \\ 4u_i^{(k,r)} + 1 & (i = n) \end{cases}, \tag{23}$$

Then by (20)-(23), for $(\forall k \geq 0)$

$$u_i^{(k,1)} = \begin{cases} 2(4^{(k-1)}) & (i = 1, ..., n-1) \\ \frac{10}{3}(4^{(k-1)}) - \frac{1}{3} & (i = n) \end{cases}, \tag{24}$$

$$u_i^{(k,2)} = \begin{cases} \frac{10}{3}(4^{(k-1)}) - \frac{1}{3} & (i = n) \\ 3(4^{(k-1)}) & (i = n-1, ..., 2) \\ 4(4^{(k-1)}) & (i = 1) \end{cases}, \tag{25}$$

and

$$u_i^{(k,3)} = \begin{cases} 4(4^{(k-1)}) & (i = 1, ..., n-1) \\ \frac{16}{3}(4^{(k-1)}) - \frac{1}{3} & (i = n) \end{cases}. \tag{26}$$

Suppose, without loss of generality, that

$$w_i^{(0,0)} \leq h < 1 \;\; (i = 1, ..., n). \tag{27}$$

Then by inductive argument it follows from (15)-(27) that for $(i = 1, ..., n)$ $(k \geq 0)$

$$w_i^{(k,1)} \leq h^{u_i^{(k+1,1)}},$$
$$w_i^{(k,2)} \leq h^{u_i^{(k+1,2)}},$$

and

$$w_i^{(k,3)} \leq h^{u_i^{(k+1,3)}},$$

whence by (27) and (14g), for $(\forall k \geq 0)$

$$w_i^{(k+1)} \leq h^{4^{(k-1)}} \quad (i = 1, ..., n).$$

So, by (17)-(27), for $(\forall k \geq 0)$

$$w(X_i^{(k)}) \leq \left(\frac{\beta}{\alpha}\right) h^{4^{(k)}} \quad (i = 1, ..., n). \tag{28}$$

Let

$$w^{(k)} = \max_{1 \leq i \leq n} \left\{ w(X_i^{(k)}) \right\}.$$

Then, by (28),

$$w^{(k)} \leq \left(\frac{\beta}{\alpha}\right) h^{4^k} \quad (\forall k \geq 0).$$

So,

$$\begin{aligned} R_4(w^{(k)}) &= \lim_{k\to\infty} sup \left\{ (w^{(k)})^{\frac{1}{(4^k)}} \right\} \\ &= \lim_{k\to\infty} \left\{ \left(\frac{\beta}{\alpha}\right)^{\frac{1}{(4^k)}} h \right\} \\ &= h \\ &< 1. \end{aligned}$$

Therefore, the R-order of convergence of *IZSS1* defined by (14) is at least 4 or

$$O_R(IZSS1, x_i^*) \geq 4 \; (i = 1, ..., n). \blacksquare$$

4 Numerical Results

In order to show the significant of the *IZSS1* method, we use the MATLAB R2007a in co-operated with the Intlab V5.5 toolbox for interval arithmetic developed by S.M. Rump [?] and it is tested on five test polynomials. The stopping criterion used is $w^{(k)} \leq 10^{-12}$.

4.1 Test Polynomials

Test Polynomial 1 : [?]

The polynomial is given by

$$p(x) = (x - \sqrt{2})(x - 3.4)(x - 5.2)(x - 7.1)$$

with

$$n = 4,$$
$$x_1^* = \sqrt{2},\ \ x_2^* = 3.4,\ \ x_3^* = 5.2,\ \ x_4^* = 7.1,$$

and the initial intervals:

$$X_1^{(0)} = [0.9, 2.1]\ ;\ X_2^{(0)} = [2.9, 3.9]\ ;\ X_3^{(0)} = [4.9, 6.3]\ ;\ X_4^{(0)} = [6.6, 8.1].$$

Test Polynomial 2 : [?]

The polynomial is given by

$$p(x) = (x - \sqrt{3})(x - \sqrt{11})(x - \sqrt{30})(x + \sqrt{3})(x + \sqrt{11})(x + \sqrt{30})$$

with

$$n = 6,$$
$$x_1^* = \sqrt{3},\ \ x_2^* = \sqrt{11},\ \ x_3^* = \sqrt{30},\ \ x_4^* = -\sqrt{3},\ \ x_5^* = -\sqrt{11},\ \ x_6^* = -\sqrt{30},$$

and the initial intervals:

$$X_1^{(0)} = [1, 2]\ ;\ X_2^{(0)} = [3, 4]\ ;\ X_3^{(0)} = [5, 6]\ ;$$
$$X_4^{(0)} = [-2, -1]\ ;\ X_5^{(0)} = [-4, -3]\ ;\ X_6^{(0)} = [-6, -5].$$

Test Polynomial 3 : [?]

The characteristic polynomial

$$p(\lambda) = det(\lambda I - A), \tag{29a}$$

where

$$A = \begin{pmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & \ddots & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & \ddots & a_{n-1} & b_{n-1} \\ & & & b_{n-1} & a_n \end{pmatrix}$$

and

$$\begin{aligned} & f^{(0)}(\lambda) = 1, \\ & f^{(1)}(\lambda) = (\lambda - a_1), \\ & f^{(k)}(\lambda) = (\lambda - a_k) f^{(k-1)}(\lambda) - (b_{k-1})^2 f^{(k-2)}(\lambda) \quad (2 \le k \le n), \\ & p(\lambda) = f^{(n)}(\lambda). \end{aligned} \tag{29b}$$

For this polynomial (Alefeld and Herzberger [?]):

$$\begin{aligned} & n = 9, \\ & b_i = 1 \ \ (i = 1, ..., n-1), \\ & a_1 = -15, \ \ a_2 = -10, \ \ a_3 = -7, \ \ a_4 = -4 \\ & a_5 = 0, \ \ a_6 = 4, \ \ a_7 = 7, \ \ a_8 = 10, \ \ a_9 = 15. \end{aligned}$$

Initial intervals:

$$\begin{aligned} & X_1^{(0)} = [-17.2, 13.8] \ \ ; \ \ X_2^{(0)} = [-12.1, -8.9] \ \ ; \ \ X_3^{(0)} = [-8.7, -6.1] \ \ ; \\ & X_4^{(0)} = [-6.0, -2.1] \ \ ; \ \ X_5^{(0)} = [-2.0, 2.3] \ \ ; \ \ X_6^{(0)} = [2.4, 6.1] \\ & X_7^{(0)} = [6.3, 8.9] \ \ ; \ \ X_8^{(0)} = [9.1, 12.9] \ \ ; \ \ X_9^{(0)} = [13.1, 17.2]. \end{aligned}$$

Test Polynomial 4 : [?]

The polynomial is given by (29) with

$$\begin{aligned} & n = 5, \\ & b_i = 1 \ \ (i = 1, ..., 4), \\ & a_1 = 0, \ \ a_2 = 3, \ \ a_3 = 6, \ \ a_4 = 9, \ \ a_5 = 12. \end{aligned}$$

Initial intervals:

$$\begin{aligned} & X_1^{(0)} = [-2.5, 2.1] \ ; \ X_2^{(0)} = [2.2, 4.5] \ ; \ X_3^{(0)} = [4.6, 7.9] \ ; \\ & X_4^{(0)} = [8.0, 10.8] \ ; \ X_5^{(0)} = [10.9, 13.1]. \end{aligned}$$

Test Polynomial 5 : [?]

The polynomial is given by (29) with

$$n = 6,$$
$$b_i = 1 \quad (i = 1, ..., 5),$$
$$a_1 = 35, \quad a_2 = 27, \quad a_3 = 21, \quad a_4 = 16, \quad a_5 = 9, \quad a_6 = 5.$$

Initial intervals:

$$X_1^{(0)} = [30, 40] \ ; \ X_2^{(0)} = [25, 29] \ ; \ X_3^{(0)} = [20, 24] \ ;$$
$$X_4^{(0)} = [13, 19] \ ; \ X_5^{(0)} = [7, 12] \ ; \ X_6^{(0)} = [3, 6].$$

4.2 Results and Discussions

The following tables summarize the results of all test polynomials

Table 1: Number of iterations and CPU times

Test Polynomial	n	*ISS1*		*IZSS1*	
		No. of iterations k	CPU times	No. of iterations k	CPU times
1	4	3	0.140625	2	0.134375
2	6	3	0.190625	2	0.181250
3	9	3	0.340625	3	0.337500
4	5	3	0.165625	2	0.159375
5	6	3	0.203125	2	0.184375

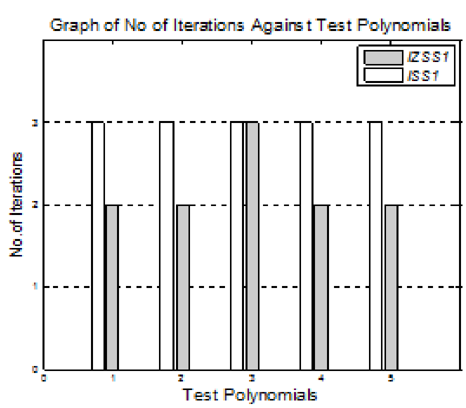

Figure 1:

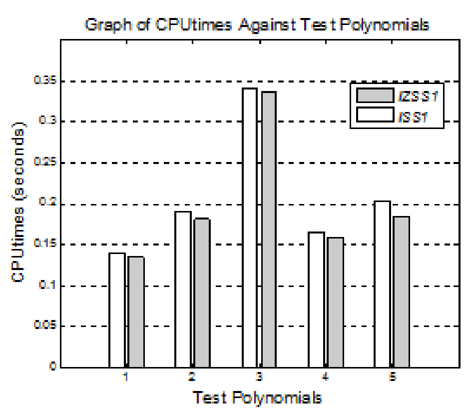

Figure 2:

Table 2: Final intervals of each component, i. (Test polynomial 5)

n	***ISS1*** Iteration $k=3$	Final Intervals	Interval widths	***IZSS1*** Iteration $k=2$	Final Intervals	Interval widths
1	$X_1^{(3)}$	[35.12417560751099, 35.12417560751299]	1.9966e-12	$X_1^{(2)}$	[35.12417560751200, 35.12417560751204]	4.2632e-14
2	$X_2^{(3)}$	[27.04236377319058, 27.04236377319060]	3.5527e-15	$X_2^{(2)}$	[27.04236377319061, 27.04236377319115]	5.1869e-13
3	$X_3^{(3)}$	[21.03245651490287 , 21.03245651490289]	3.5527e-15	$X_3^{(2)}$	[21.03245651490139, 21.03245651490141]	1.0658e-14
4	$X_4^{(3)}$	[15.94449367035688, 15.94449367035689]	1.7763e-15	$X_4^{(2)}$	[15.94449367035731, 15.94449367035732]	3.5527e-15
5	$X_5^{(3)}$	[9.09739136898243, 9.09739136898244]	1.7763e-15	$X_5^{(2)}$	[9.09739136898250, 9.09739136898251]	1.7763e-15
6	$X_6^{(3)}$	[4.75911906505652, 4.75911906505653]	9.7699e-15	$X_6^{(2)}$	[4.75911906505600, 4.75911906505714]	1.1297e-12

Table 3: The width of the final intervals ($w_i^{(k)} = w(X_i^{(k)})$) at iteration k.(Test polynomial 3)

Iteration k	**Steps**	**Interval Width** $w_i^{(k)}$ **of** $X_i^{(k)}$ Method ***ISS1***		Method ***IZSS1***	
1	**Step 1 (15d)**	$w_1^{(1)}$	0.929835	$w_1^{(1)}$	0.929835
		$w_2^{(1)}$	1.483985	$w_2^{(1)}$	1.483985
		$w_3^{(1)}$	1.195206	$w_3^{(1)}$	1.195206
		$w_4^{(1)}$	0.755458	$w_4^{(1)}$	0.755458
		$w_5^{(1)}$	0.545575	$w_5^{(1)}$	0.545575
		$w_6^{(1)}$	0.820056	$w_6^{(1)}$	0.820056
		$w_7^{(1)}$	1.124704	$w_7^{(1)}$	1.124704
		$w_8^{(1)}$	1.474032	$w_8^{(1)}$	1.474032
		$w_9^{(1)}$	0.033090	$w_9^{(1)}$	0.033090
	Step 2 (15e)	$w_1^{(1)}$	0.055474	$w_1^{(1)}$	0.055474
		$w_2^{(1)}$	0.183665	$w_2^{(1)}$	0.183665
		$w_3^{(1)}$	0.549351	$w_3^{(1)}$	0.549351
		$w_4^{(1)}$	0.177100	$w_4^{(1)}$	0.177100
		$w_5^{(1)}$	0.158262	$w_5^{(1)}$	0.158262
		$w_6^{(1)}$	0.302859	$w_6^{(1)}$	0.302859
		$w_7^{(1)}$	0.526478	$w_7^{(1)}$	0.526478
		$w_8^{(1)}$	0.598351	$w_8^{(1)}$	0.598351
		$w_9^{(1)}$	0.033090	$w_9^{(1)}$	0.033090
	Step 3 (15f)	(Not Applicable)		$w_1^{(1)}$	0.055474
				$w_2^{(1)}$	0.100158
				$w_3^{(1)}$	0.085618
				$w_4^{(1)}$	0.029657
				$w_5^{(1)}$	0.039940
				$w_6^{(1)}$	0.116031
				$w_7^{(1)}$	0.184678
				$w_8^{(1)}$	0.075224
				$w_9^{(1)}$	0.002900

Table 4: *

Table 3:(continue).

Iteration k	Steps	Interval Width $w_i^{(k)}$ of $X_i^{(k)}$ Method *ISS1*		Method *IZSS1*	
2	**Step 1 (15d)**	$w_1^{(2)}$	8.110418e-04	$w_1^{(2)}$	2.287748e-04
		$w_2^{(2)}$	0.005333	$w_2^{(2)}$	7.744356e-04
		$w_3^{(2)}$	0.039052	$w_3^{(2)}$	1.106855e-04
		$w_4^{(2)}$	0.012682	$w_4^{(2)}$	2.574670e-04
		$w_5^{(2)}$	0.012983	$w_5^{(2)}$	4.822782e-04
		$w_6^{(2)}$	0.021760	$w_6^{(2)}$	0.001491
		$w_7^{(2)}$	0.018609	$w_7^{(2)}$	5.228205e-04
		$w_8^{(2)}$	0.001199	$w_8^{(2)}$	7.810209e-06
		$w_9^{(2)}$	1.234489e-05	$w_9^{(2)}$	4.249435e-08
	Step 2 (15e)	$w_1^{(2)}$	1.288216e-06	$w_1^{(2)}$	6.014025e-09
		$w_2^{(2)}$	1.278656e-05	$w_2^{(2)}$	6.191749e-07
		$w_3^{(2)}$	4.701623e-04	$w_3^{(2)}$	6.962070e-07
		$w_4^{(2)}$	8.915411e-04	$w_4^{(2)}$	9.826203e-07
		$w_5^{(2)}$	5.516567e-04	$w_5^{(2)}$	1.323301e-06
		$w_6^{(2)}$	7.088768e-04	$w_6^{(2)}$	4.881481e-06
		$w_7^{(2)}$	0.001203	$w_7^{(2)}$	1.360880e-05
		$w_8^{(2)}$	8.276142e-04	$w_8^{(2)}$	3.737551e-06
		$w_9^{(2)}$	1.234489e-05	$w_9^{(2)}$	4.249435e-08
	Step 3 (15f)	(Not Applicable)		$w_1^{(1)}$	6.014025e-09
				$w_2^{(2)}$	2.414903e-08
				$w_3^{(2)}$	5.412288e-09
				$w_4^{(2)}$	1.331421e-08
				$w_5^{(2)}$	2.667564e-08
				$w_6^{(2)}$	1.021968e-07
				$w_7^{(2)}$	2.600542e-08
				$w_8^{(2)}$	2.731806e-10
				$w_9^{(2)}$	2.399858e-12
3	**Step 1 (15d)**	$w_1^{(3)}$	4.099832e-12	$w_1^{(3)}$	1.776357e-15
		$w_2^{(3)}$	9.765522e-11	$w_2^{(3)}$	1.776357e-15
		$w_3^{(3)}$	2.408912e-08	$w_3^{(3)}$	8.881784e-16
		$w_4^{(3)}$	9.976478e-09	$w_4^{(3)}$	4.440892e-16
		$w_5^{(3)}$	2.153843e-08	$w_5^{(3)}$	4.946036e-17
		$w_6^{(3)}$	6.283580e-09	$w_6^{(3)}$	4.440892e-16
		$w_7^{(3)}$	3.167606e-08	$w_7^{(3)}$	8.881784e-16
		$w_8^{(3)}$	2.126850e-10	$w_8^{(3)}$	1.776357e-15
		$w_9^{(3)}$	1.065814e-14	$w_9^{(3)}$	2.399858e-12
	Step 2 (15e)	$w_1^{(3)}$	4.099832e-12	(Converged)	
		$w_2^{(3)}$	1.776357e-15		
		$w_3^{(3)}$	2.664535e-15		
		$w_4^{(3)}$	1.953993e-13		
		$w_5^{(3)}$	2.967427e-13		
		$w_6^{(3)}$	1.083578e-13		
		$w_7^{(3)}$	9.245937e-13		
		$w_8^{(3)}$	1.335820e-12		
		$w_9^{(3)}$	1.065814e-14		
	Step 3 (15f)	(Not Applicable)			

Table 5: The width of final intervals at the iteration k.(Test polynomial 4)

Iteration	**Interval Width**	**Methods**	
k	$w_i^{(k)}$	***ISS1***	***IZSS1***
1	$w_1^{(1)}$	0.005866	0.005866
	$w_2^{(1)}$	0.044377	0.025062
	$w_3^{(1)}$	0.087971	0.017840
	$w_4^{(1)}$	0.123660	0.018660
	$w_5^{(1)}$	0.111359	0.003951
2	$w_1^{(2)}$	5.730180e-10	1.009359e-12
	$w_2^{(2)}$	3.820871e-08	4.779732e-12
	$w_3^{(2)}$	1.475231e-06	7.931433e-13
	$w_4^{(2)}$	5.469630e-06	8.526513e-13
	$w_5^{(2)}$	7.291283e-06	3.552714e-15
3	$w_1^{(3)}$	1.110223e-16	(Converged)
	$w_2^{(3)}$	5.773160e-15	
	$w_3^{(3)}$	3.277378e-13	
	$w_4^{(3)}$	3.677059e-13	
	$w_5^{(3)}$	1.776357e-15	

Table 1 shows the comparison of the number of iterations and the CPU times in seconds, between the procedures *ISS1* and *IZSS1* while Figure 1 and Figure 2 are the graphs of the number of iterations and the CPU times respectively. For Table 2, we take the results of test polynomial 5 to show the final intervals of the procedure *IZSS1* compare to the procedure *ISS1* including their widths before the algorithm stop at the stopping criterion,$w^{(k)} \leq 10^{-12}$. Table 3 is a result of the final interval width of test polynomial 3. It shows that even though the numbers of iterations of both procedures are the same, the algorithm *IZSS1* converge earlier by one step. We observe in Table 4 that for test polynomial 4, the method *IZSS1* is better than the method *ISS1* in term of number of iterations.

5 Conclusion

We have shown analytically that the interval symmetric single-step procedure *IZSS1* gives higher rate of convergence, where the R-order of convergence of *IZSS1* is at least 4 or $O_R(IZSS1, x_i^*) \geq 4$. While the R-order of convergence of *ISS1* (Monsi [**?**]) is at least 3, that is $O_R(ISS1, x_i^*) \geq 3$. It is clear from Table 1, Table 2, Table 3 and Table 4 that the procedure *IZSS1* numerically requires less CPU times and number of iterations and furthermore the final intervals of *IZSS1* have a better accuracy than does ISS1 where the stopping criterion used is $w^{(k)} \leq 10^{-12}$.

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