The RHFs for Solution of Nonlinear Fredholm
Integro-Differential Equations

Farshid Mirzaee

Department of Mathematics
Faculty of Science, Malayer University
Malayer, 65719-95863, Iran
f.mirzaee @ malayeru.ac.ir
mirzaee@mail.iust.ac.ir

Abstract

The aim of the present paper is to introduce a numerical method for solving nonlinear Fredholm integro-differential equations of the second kind. The main idea is based on the rationalized Haar functions (RHFs) method. We reduce the nonlinear Fredholm integro-differential equation to a system of nonlinear equations. The examples that illustrate the pertinent features of the method are presented.

Keywords: RHFs, Nonlinear Fredholm integro-differential equations, Operational matrix, Product operation

1 Introduction

The solution of integro-differential equations have a major role in the fields of science and engineering. When a physical system is modeled under the differential sense, it finally gives a differential equation, an integral equation or an integro-differential equation. Forthcoming of the first two equations mostly appear in the last equation. There are various techniques for solving an integro-differential equation [1, 2, 6, 10].
In this paper, we present a RHFs method for solving nonlinear Fredholm integro-differential equations. Several numerical methods for approximating the solution of this kind of integral equations are known, and many different basic functions have been used, such as orthogonal bases and wavelets. In the present paper, we introduce a direct computational method for solving nonlinear Fredholm integro-differential equations. We apply RHFs to solve the nonlinear Fredholm integro-differential equations.

2 Properties of RHFs

2.1 Definition of RHFs

The RHFs RH \((r, t)\), \(r = 1, 2, 3, \ldots\), are composed of three values 1, −1 and 0 and can be defined on the interval \([0, 1)\) as

\[
RH(r, t) = \begin{cases} 
1, & J_1 \leq t < J_{\frac{1}{2}} \\
-1, & J_{\frac{1}{2}} \leq t < J_0, \\
0, & \text{otherwise}
\end{cases}
\]  

(1)

where \(j_u = \frac{j - u}{2^i}\) and \(u = 0, \frac{1}{2}, 1.\ [7]\).

The value of \(r\) is defined by two parameters \(i\) and \(j\) as

\[
r = 2^i + j - 1, \quad i = 0, 1, 2, \ldots, \quad j = 1, 2, 3, \ldots, 2^i,
\]  

(2)

RH(0, t) is defined for \(i = j = 0\) and is given by

\[
RH(0, t) = 1; \quad 0 \leq t < 1.
\]

The orthogonality property is given by

\[
\langle RH(r, t), RH(v, t) \rangle = \int_0^1 RH(r, t)RH(v, t)dt = \begin{cases} 
2^{-i}, & \text{for } r = v \\
0, & \text{for } r \neq v,
\end{cases}
\]
where \(v\) and \(r\) introduced in equation (2).

### 2.2 Function approximation

A function \(f(t)\) defined over the interval \(L^2[0,1]\) may be expanded in RHF as

\[
f(t) = \sum_{r=0}^{\infty} a_r RH(r,t), \quad (3)
\]

where

\[
a_r = \frac{\langle f(t), RH(r,t) \rangle}{\langle RH(r,t), RH(r,t) \rangle}. \quad (4)
\]

If we let \(i = 0, 1, 2, 3, \cdots, \alpha\) then the infinite series in equation (3) is truncated up to its first \(k\) terms as

\[
f(t) \simeq \sum_{r=0}^{k-1} a_r RH(r,t) = A^T \phi(t), \quad (5)
\]

where \(k = 2^{\alpha+1}, \alpha = 0, 1, 2, \ldots\), \( \phi_r(t) = RH(r,t)\),

\[
A = [a_0, a_1, \ldots, a_{k-1}]^T, \quad (6)
\]

\[
\phi(t) = [\phi_0(t), \phi_1(t), \ldots, \phi_{k-1}(t)]^T. \quad (7)
\]

If each waveform is divided into \(k\) intervals, the magnitude of the waveform can be represented as

\[
\hat{\phi}_{k \times k} = [\phi(\frac{1}{2k}), \phi(\frac{3}{2k}), \ldots, \phi(\frac{2k-1}{2k})], \quad (8)
\]

where in equation (8) the row denotes the order of the RHF [3]. By using
equation (8) and (5) we get
\[
\begin{bmatrix}
 f\left(\frac{1}{2k}\right), f\left(\frac{3}{2k}\right), \ldots, f\left(\frac{2k-1}{2k}\right)
\end{bmatrix} = A^T \hat{\phi}_{k \times k} .
\] (9)

We can also approximate the function \( k(t, s) \in L^2([0, 1] \times [0, 1]) \) as follows
\[
k(t, s) \simeq \phi_T^T(t) H \phi(s) ,
\] (10)

where \( H = [h_{ij}]_{k \times k} \) is an \( k \times k \) matrix that:
\[
h_{ij} = \frac{\langle RH(i, t), \langle k(t, s), RH(j, s) \rangle \rangle}{\langle RH(i, t), RH(i, t) \rangle \langle RH(j, t), RH(j, t) \rangle} ,
\] (11)

for \( i, j = 0, 1, 2, \ldots, k-1 \). From equation (8) and (9) we have
\[
H = \left( \hat{\phi}_{k \times k}^{-1} \right)^T \hat{H} \hat{\phi}_{k \times k}^{-1} ,
\] (12)

where
\[
\hat{H} = [\hat{h}_{ij}]_{k \times k} \quad \hat{h}_{ij} = k\left(\frac{2i - 1}{2k}, \frac{2j - 1}{2k}\right) , \quad i, j = 1, 2, \ldots, k ,
\]

and
\[
\hat{\phi}_{k \times k}^{-1} = \left( \frac{1}{k} \right) \hat{\phi}_{k \times k}^T \int_0^1 \phi(t) \phi(t)^T dt \right)^{-1} .
\] (13)

We also define the matrix \( D \) as follows
\[
D = \int_0^1 \phi(t) \phi(t)^T dt .
\] (14)

For the RHFs, \( D \) has the following form \([5,9]\):
\[ D = \text{diag}(1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) .\]

### 2.3 Operational matrix of integration

The integration of the \( \phi(t) \) defined in equation (7) is given by

\[
\int_0^t \phi(t')dt' \simeq P\phi(t) ,
\]

where \( P = P_{k\times k} \) is the \( k \times k \) operational matrix for integration and is given in [5] as

\[
P_{k\times k} = \frac{1}{2k} \begin{bmatrix}
2kP_{(\frac{1}{2})\times(\frac{1}{2})} & -\hat{\phi}_{(\frac{1}{2})\times(\frac{1}{2})} \\
\hat{\phi}^{-1}_{(\frac{1}{2})\times(\frac{1}{2})} & 0
\end{bmatrix} ,
\]

where \( \hat{\phi}_{1\times1} = [1] \quad P_{1\times1} = [\frac{1}{2}] \).

### 2.4 The product operation matrix

The product operation matrix for RHF is defined as follows :

\[
\phi(t)\phi^T(t)A \simeq \tilde{A}_{k\times k}\phi(t) ,
\]

where A is given in equation (6) and \( \tilde{A}_{k\times k} \) in an \( k \times k \) matrix , which is called the product operation matrix of RHF.

In general we have

\[
\tilde{A}_{k\times k} = \begin{bmatrix}
\tilde{A}_{(\frac{1}{2})\times(\frac{1}{2})} & \tilde{H}_{(\frac{1}{2})\times(\frac{1}{2})} \\
\tilde{H}_{(\frac{1}{2})\times(\frac{1}{2})} & \tilde{D}_{(\frac{1}{2})\times(\frac{1}{2})}
\end{bmatrix} ,
\]
3458

whit

\[ \hat{A}_{1 \times 1} = a_0 , \]

\[ \hat{H}_{(\frac{k}{2}) \times (\frac{k}{2})} = \hat{\phi}_{(\frac{k}{2}) \times (\frac{k}{2})} \cdot \text{diag}\left[ a_{\frac{k}{2}}, a_{\frac{k}{2}+1}, \ldots, a_{k-1} \right] , \]

\[ \hat{H}_{(\frac{k}{2}) \times (\frac{k}{2})} = \text{diag}\left[ a_{\frac{k}{2}}, a_{\frac{k}{2}+1}, \ldots, a_{k-1} \right] \cdot \hat{\phi}^{-1}_{(\frac{k}{2}) \times (\frac{k}{2})} , \]

and

\[ \hat{D}_{(\frac{k}{2}) \times (\frac{k}{2})} = \text{diag}\left[ a_{0}, a_{1}, \ldots, a_{\frac{k}{2}-1} \right] \cdot \hat{\phi}_{(\frac{k}{2}) \times (\frac{k}{2})} \]

See [9].

3 Nonlinear Fredholm integro-differential equations

We consider the following nonlinear Fredholm integro-differential equations:

\[
\begin{cases}
  y'(t) = x(t) + \int_0^1 k(t, s)F(y(t))ds , & t \in [0, 1] \\
  y(0) = y_0 ,
\end{cases}
\]

(19)

where \( x(t), F(y(t)) \in L^2[0, 1), k(t, s) \in L^2([0, 1] \times [0, 1]) \) and \( y(t) \) unknown function with \( F(y(t)) \) nonlinear in \( y(t) \). See [3,4]. A general formula or \( F(y(t)) \) can be written as

\[ w(t) = F(y(t)) , \quad t \in [0, 1] . \]

(20)
Form equation \( y(t) = \int_0^t y'(t)dt' + y(0) \), and (20) we get

\[
w(t) = F\left( \int_0^t y'(t)dt' + y(0) \right), \tag{21}
\]

with substituting equation (19) in equation (21) we get

\[
w(t) = F\left( \int_0^t x(t')dt' + \int_0^t \int_0^1 k(t',s)w(s)dsdt' + y(0) \right). \tag{22}
\]

We approximate \( w(t), x(t) \) and \( y(0) \) by equation (5) as follows :

\[
w(t) \simeq W^T \phi(t), \tag{23}
\]

\[
x(t) \simeq X^T \phi(t), \tag{24}
\]

\[
y(0) \simeq Y_0^T \phi(t), \tag{25}
\]

where \( W, X, Y_0 \) and \( \phi(t) \) are given in equations (6) and (7) respectively. Using equations (10),(17),(23),(24) and (25) we have

\[
\int_0^t x(t')dt' + \int_0^t \int_0^1 k(t',s)w(s)dsdt' + y(0) = X^T P \phi(t) + \phi^T(t)P^T \bar{W}e_1 + Y_0^T \phi(t).
\]

(26)

With substituting in equation (22) we get

\[
w(t) = F(X^T P \phi(t) + \phi^T(t)P^T \bar{W}e_1 + Y_0^T \phi(t)). \tag{27}
\]

where
\[ e_1 = [1, 0, 0, \ldots, 0]^T \in \mathbb{R}^k. \]

In order to construct the approximations for \( w(t) \) we collocate equations (27) in \( k \) points. By using equation (8) and Newton-Cotes points given in [8] as

\[ t_p = \frac{2p - 1}{2k}, \quad p = 1, 2, 3, \ldots, k, \]

we have

\[ \phi(t_p) = \hat{\phi}_{k \times k} \cdot e_p, \quad p = 1, 2, 3, \ldots, k, \]

where

\[ e_p = [0, 0, \ldots, 0, 1, 0, \ldots, 0]^T \in \mathbb{R}^k, \]

and 1 pth-component. Equation (27) can be expressed as

\[ w(t_p) = F(X^T P \hat{\phi}_{k \times k} \cdot e_p + e_0^T \hat{\phi}_{k \times k}^T P^T H \hat{W} e_1 + Y_0^T \hat{\phi}_{k \times k} \cdot e_p); \quad p = 1, 2, 3, \ldots, k. \]

(29)

By solving nonlinear system we can find the vector \( W \). Using equations (10),(17),(19) and (20) we get

\[ y'(t) = x(t) + \hat{\phi}(t) H \hat{W} e_1. \]

(30)

Equation (30) can be solved for unknown \( y'(t) \). We approximate \( y(t) \) and \( y'(t) \) by equation (5) as follows:

\[ y(t) = Y^T \phi(t), \quad y'(t) = Y'^T \phi(t). \]
With substituting in equation:

\[ y(t) = \int_0^t y'(t')dt' + y(0) , \]

we get

\[ y(t) \simeq (Y'^TP + Y_0^T)\phi(t) . \] (31)

By solving equation (31) we can find the vector \( Y \), so

\[ y(t) \simeq Y^T\phi(t) . \]

4 Numerical examples

Example 1. Consider the nonlinear Fredholm integro-differential equation:

\[
\begin{align*}
    y'(t) &= \frac{5}{4} - \frac{1}{3}t^2 + \int_0^1 (t^2 - s)y^2(s)ds; \quad t \in [0, 1), \\
    y(0) &= 0,
\end{align*}
\]

and exact solution \( y(t) = t \), result are shown in Table 1.

<p>| Table 1 | Numerical results for Example 1 |</p>
<table>
<thead>
<tr>
<th>Nodes</th>
<th>$t$</th>
<th>Approximate for $k=16$</th>
<th>Approximate for $k=32$</th>
<th>Exact solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.00155</td>
<td>0.00001</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>0.10401</td>
<td>0.10002</td>
<td>0.1</td>
</tr>
<tr>
<td>0.2</td>
<td>0.2</td>
<td>0.20398</td>
<td>0.20008</td>
<td>0.2</td>
</tr>
<tr>
<td>0.3</td>
<td>0.3</td>
<td>0.30147</td>
<td>0.30007</td>
<td>0.3</td>
</tr>
<tr>
<td>0.4</td>
<td>0.4</td>
<td>0.40640</td>
<td>0.40008</td>
<td>0.4</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.50887</td>
<td>0.50001</td>
<td>0.5</td>
</tr>
<tr>
<td>0.6</td>
<td>0.6</td>
<td>0.60382</td>
<td>0.60001</td>
<td>0.6</td>
</tr>
<tr>
<td>0.7</td>
<td>0.7</td>
<td>0.70375</td>
<td>0.70002</td>
<td>0.7</td>
</tr>
<tr>
<td>0.8</td>
<td>0.8</td>
<td>0.79923</td>
<td>0.80008</td>
<td>0.8</td>
</tr>
<tr>
<td>0.9</td>
<td>0.9</td>
<td>0.90214</td>
<td>0.89991</td>
<td>0.9</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.99863</td>
<td>0.99992</td>
<td>1</td>
</tr>
</tbody>
</table>

Example 2. Consider the nonlinear Fredholm integro-differential equation:

$$
\begin{align*}
\frac{dy}{dt}(t) &= -\frac{1}{2}e^t + \frac{3}{2}e^t + \int_0^1 e^{t-s}y^3(s)ds ; \quad t \in [0, 1), \\
y(0) &= 1,
\end{align*}
$$

and exact solution $y(t) = e^t$, result are shown in Table 2.
Table 2 : Numerical results for Example 2

<table>
<thead>
<tr>
<th>Nodes</th>
<th>$t$</th>
<th>Approximate for $k=16$</th>
<th>Approximate for $k=32$</th>
<th>Exact solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.00165</td>
<td>1.00008</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>0.1</td>
<td>1.10721</td>
<td>1.10519</td>
<td>1.10517</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>1.22345</td>
<td>1.22145</td>
<td>1.22140</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>1.34370</td>
<td>1.34987</td>
<td>1.34986</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>1.49003</td>
<td>1.49184</td>
<td>1.49182</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>1.64692</td>
<td>1.64879</td>
<td>1.64872</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>1.82981</td>
<td>1.82212</td>
<td>1.82211</td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>2.01829</td>
<td>2.01373</td>
<td>2.01376</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>2.22349</td>
<td>2.22552</td>
<td>2.22554</td>
<td></td>
</tr>
<tr>
<td>0.9</td>
<td>2.45438</td>
<td>2.45969</td>
<td>2.45960</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2.71993</td>
<td>2.71821</td>
<td>2.71828</td>
<td></td>
</tr>
</tbody>
</table>

5 Conclusion

Nonlinear integro-differential equations are usually difficult to solve analytically. In many cases, it is required to obtain the approximate solutions, for this purpose the presented method can be proposed. In the presented method we approximate the nonlinear part of the integro-differential equation with the RHFs. A quadrature formula is used to compute the coefficients of expansion of any given function. This method can be extended and applied to the high-order nonlinear integro-differential equations, system of nonlinear integro-differential equations, but some modifications are required.
References


Received: April, 2011