Necessary and Sufficient Condition for Schur Convexity of the Two-Parameter Symmetric Homogeneous Means

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Abstract. An necessary and sufficient condition for Schur convexity of the two-parameter symmetric homogeneous means is given, which improves Witkowski’s result. As an application, Schur convexity of the four-parameter homogeneous means is perfectly solved. This improves and generalizes the known results for the Schur convexity of Stolarsky means and Gini means.

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1. Introduction

To begin with, we recall the definition of Schur convex functions.

Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a set with nonempty interior. Then $\phi : \Omega \to \mathbb{R}$ is called Schur convex on $\Omega$ if $\phi(x) \leq \phi(y)$ for each two $n$-tuples $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ of $\Omega$, such that $x \prec y$ holds. The relationship of majorization $x \prec y$ means that

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \quad \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

where $1 \leq k \leq n - 1$, and $x_{[i]}$ denotes the $i$-th largest component of $x$. $\phi$ is called Schur concave if $-\phi$ is Schur convex.

The following well-known result was proved by Marshall and Olkin [5].

Theorem M-O. Let $\Omega \subset \mathbb{R}^2$ be a symmetric convex set with nonempty interior $\Omega$ and $\phi : \Omega \to \mathbb{R}$ be a continuous and symmetric function on $\Omega$. If $\phi$ is differentiable on $\Omega$, then $\phi$ is Schur convex (Schur concave) on $\Omega$ if and only
if
\[(y - x) \left( \frac{\partial \phi}{\partial y} - \frac{\partial \phi}{\partial x} \right) > (\leq) 0\]
for all \((x, y) \in \Omega\) with \(x \neq y\).

Suppose \(p, q \in \mathbb{R}\) and \(a, b \in \mathbb{R}_+ = (0, \infty)\). For \(a \neq b\) the Stolarsky means are defined as
\[
S_{p,q}(a, b) = \begin{cases} 
\left( \frac{q a^p - b^p}{p a^q - b^q} \right)^{1/(p-q)} & \text{if } pq(p - q) \neq 0, \\
\left( \frac{1}{p} \frac{a^p - b^p}{p \ln a - \ln b} \right)^{1/p} & \text{if } p \neq 0, q = 0, \\
\left( \frac{1}{q} \frac{a^q - b^q}{q \ln a - \ln b} \right)^{1/q} & \text{if } q \neq 0, p = 0, \\
\exp \left( \frac{a^p \ln a - b^p \ln b}{a^p - b^p} - \frac{1}{p} \right) & \text{if } p = q \neq 0, \\
\sqrt{ab} & \text{if } p = q = 0,
\end{cases}
\]
and \(S_{p,q}(a, a) = a\) (see [11]). It follows from (1.2) that \(S_{1,0}(a, b) = L(a, b)\) –the logarithmic mean, \(S_{1,1}(a, b) = I(a, b)\) –the identric (exponential) mean, \(S_{2,1}(a, b) = A(a, b)\) –the arithmetic mean, \(S_{3/2,1/2}(a, b) = He(a, b)\) –Heronian mean, etc.

Another two-parameter family of means was introduced by C. Gini in [3]. That are defined as
\[
G_{p,q}(a, b) = \begin{cases} 
\left( \frac{a^p + b^p}{a^q + b^q} \right)^{1/(p-q)} & \text{if } p \neq q, \\
\exp \left( \frac{a^p \ln a + b^p \ln b}{a^p + b^p} \right) & \text{if } p = q.
\end{cases}
\]

The Schur convexities of \(S_{p,q}(a, b)\) and \(G_{p,q}(a, b)\) with respect to \((a, b)\) were investigated by Qi [8], Shi [10], Li [4], Chu, Zhang [1] et al. Until now, they have been perfectly solved by Chu and Zhang [1], Wang and Zhang [12, 14], respectively. Recently, Chu and Xia were also proved the same result as Wang and Zhang’s [2].

The Schur convexity of \(S_{p,q}(a, b)\) with respect to \((p, q)\) was investigated by Qi [7], who first obtained the following result.

**Theorem Q.** For fixed \(a, b > 0\) with \(a \neq b\), the Stolarsky means \(S_{p,q}(a, b)\) are Schur concave on \([0, \infty) \times [0, \infty)\) and Schur convex on \((-\infty, 0) \times (-\infty, 0)\) with respect to \((p, q)\).

Sándor [9] researched the Schur convexity of the Gini mean values \(G_{p,q}(a, b)\) with respect to \((p, q)\) and obtained a similar result.

**Theorem S.** For fixed \(a, b > 0\) with \(a \neq b\), the Gini mean values \(G_{p,q}(a, b)\) are Schur concave on \([0, \infty) \times [0, \infty)\) and Schur convex on \((-\infty, 0) \times (-\infty, 0)\) with respect to \((p, q)\).
In 2005, the Stolarsky and Gini means were generalized to the following two-parameter homogeneous function by Yang [17].

**Definition 1.** Let \( f: \mathbb{R}^2_+ \rightarrow \mathbb{R}_+ \) is a homogeneous, continuous function and has first partial derivatives. Then the function \( \mathcal{H}_f: \mathbb{R}^2 \times \mathbb{R}^2_+ \rightarrow \mathbb{R}_+ \) is called a homogeneous function generated by \( f \) with parameters \( p \) and \( q \) if \( \mathcal{H}_f \) is defined by

\[
(1.4) \mathcal{H}_f(p, q; x, y) = \left( \frac{f(x^p, y^p)}{f(x^q, y^q)} \right)^{1/(p-q)} \text{ if } p \neq q,
\]

\[
(1.5) \mathcal{H}_f(p, p; x, y) = \exp \left( \frac{x^p f_x(x^p, y^p) \ln x + y^p f_y(x^p, y^p) \ln y}{f(x^p, y^p)} \right) \text{ if } p = q,
\]

where \( f_x(x, y) \) and \( f_y(x, y) \) denote partial derivative with respect to first and second variable of \( f(x, y) \), respectively.

\( \mathcal{H}_f(p, q; x, y) \) is also called two-parameter homogeneous function for short, and usually simply denotes by \( \mathcal{H}_f(p, q) \) or \( \mathcal{H}_f(x, y) \).

**Remark 1.** Witkowski [13] proved that if \( f(x, y) \) is a symmetric and 1-order homogeneous function, then for all \( p, q \) \( \mathcal{H}_f(p, q; x, y) \) is a mean of positive numbers \( x \) and \( y \) if and only if \( f(x, y) \) is increasing in both variables on \( \mathbb{R}_+ \). In fact, it is easy to see that the condition "\( f(x, y) \) is symmetry" can be removed. If \( \mathcal{H}_f(p, q; x, y) \) is a mean of positive numbers \( x \) and \( y \) then it is called two-parameter homogeneous mean generated by \( f \).

The two-parameter homogeneous function \( \mathcal{H}_f(p, q; x, y) \) generated by \( f \) is very important because it can generates many well-known means. For examples, substituting \( L(x, y), A(x, y), I(x, y), He(x, y) \) for \( f(x, y) \) in Definition 1, we can obtain the two-parameter logarithmic means \( \mathcal{H}_L(p, q; x, y) = S_{p,q}(x, y) \), two-parameter arithmetic means \( \mathcal{H}_A(p, q; x, y) = G_{p,q}(x, y) \), two-parameter identric means \( I_{p,q}(x, y) := \mathcal{H}_I(p, q; x, y) \), two-parameter Heronian means \( He_{p,q}(x, y) := \mathcal{H}_{He}(p, q; x, y) \).

In 2009, Witkowskt used Merkle’s results [6] to prove the following.

**Theorem W.** The following conditions are equivalent:

(a) For all \( p, q \geq 0 \) and all \( x, y > 0 \ln \mathcal{H}_f \) is convex (concave) in \( p \) and \( q \).
(b) For all \( p, q \geq 0 \) and all \( x, y > 0 \ln \mathcal{H}_f \) is Schur convex (Schur concave) in \( p \) and \( q \).
(c) \( \hat{f}'(t) \) is convex (concave) for \( t \geq 0 \), where \( \hat{f}(t) = \ln f(e^t, 1) \).
(d) For all \( p, q \leq 0 \) and all \( x, y > 0 \ln \mathcal{H}_f \) is convex (concave) in \( p \) and \( q \).
(e) For all \( p, q \leq 0 \) and all \( x, y > 0 \ln \mathcal{H}_f \) is Schur concave (Schur convex) in \( p \) and \( q \).
(f) \( \hat{f}'(t) \) is concave (convex) for \( t \geq 0 \), where \( \hat{f}(t) = \ln f(e^t, 1) \).

It is clear that Theorem W is a generalization of Theorem Q and S. However, the Schur convexity of \( \mathcal{H}_f(p, q; a, b) \) on other plane domain with respect to \( (p, q) \) has not been discussed.
The purpose of this work is to improve Theorem W. Our main result is as follows.

**Theorem 1.** Suppose that \( f : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is a symmetric, homogenous and three-time differentiable function. If \( J = (x - y)(xI)_x < (>)0 \), where \( I = (\ln f)_{xy} \),

\[
(1.6) \quad J = (x - y)(xI)_x < (>)0, \text{ where } I = (\ln f)_{xy},
\]

then for fixed \( a, b > 0 \) with \( a \neq b \), \( \mathcal{H}_f(p, q; a, b) \) is Schur convex if and only if \( p + q > (>)0 \) and Schur concave if and only if \( p + q < (>)0 \) with respect to \( (p, q) \).

2. **Proof of Main Result**

In order to prove our main result, we continue to adopt our notations and use straightforward differentiations.

The following function will play an important role in proof of main result:

\[
(2.1) \quad t \rightarrow T(t) := \ln f(a^t, b^t).
\]

Some properties of which read as follows:

**Property 1** ([18, (1.15), (2.10), (2.11), (2.12)]). Let \( f(x, y) \) be a positive, symmetric, \( n \)-order homogenous and two-time differentiable function defined on \( \mathbb{R}^2_+ \). Then

\[
(2.2) \quad T(t) - T(-t) = 2nt \ln G,
\]

\[
(2.3) \quad T'(t) + T'(-t) = 2n \ln G = 2T''(0),
\]

\[
(2.4) \quad T''(-t) = T''(t),
\]

where \( G = \sqrt{ab} \).

**Remark 2.** By Property 1, we see that the function \( t \rightarrow T(t) = \ln f(a^t, b^t) \) is an even function.

**Property 2** ([18, (1.12), (2.5), (2.8)]). Let \( f(x, y) \) be a positive, homogenous and two-time differentiable function defined on \( \mathbb{R}^2_+ \). Then

\[
(2.5) \quad T''(t) = -xyI \ln^2(b/a), \text{ where } I = (\ln f)_{xy},
\]

\[
(2.6) \quad T'''(t) = -Ct^{-3}J, \text{ where } J = (x - y)(xI)_x, C = \frac{xy \ln^3(x/y)}{x - y} > 0,
\]

where \( x = a^t, y = b^t \).

**Remark 3.** It follows from property 2 that

\[
(2.7) \quad \text{sgn}(I) = - \quad \text{sgn}(T''(t)),
\]

\[
(2.8) \quad \text{sgn}(J) = - \quad \text{sgn}(t) \text{sgn}(T'''(t)).
\]

The following property is also crucial in the proof of main result.
Property 3 ([18, (1.13)]). If $T'(t)$ is continuous on $[p, q]$ or $[q, p]$, then $\ln \mathcal{H}_f(p, q)$ can be expressed in integral form as

$$
(2.9) \quad \ln \mathcal{H}_f(p, q) = \begin{cases} 
\frac{1}{p - q} \int_q^p T'(t) \, dt & \text{if } p \neq q, \\
T'(q) & \text{if } p = q
\end{cases} = \int_0^1 T'(tp + (1 - t)q) \, dt.
$$

We now prove our main result.

Proof of Theorem 1. Direct partial derivative calculation for (2.9) leads to

$$
(2.10) \quad \frac{\partial \ln \mathcal{H}_f}{\partial p} = \frac{1}{\mathcal{H}_f} \frac{\partial \mathcal{H}_f}{\partial p} = \int_0^1 tT''(t_1(t)) \, dt,
$$

$$
(2.11) \quad \frac{\partial \ln \mathcal{H}_f}{\partial q} = \frac{1}{\mathcal{H}_f} \frac{\partial \mathcal{H}_f}{\partial q} = \int_0^1 (1 - t)T''(t_1(t)) \, dt,
$$

where $t_1(t) = tp + (1 - t)q$. Subtracting (2.11) from (2.10) yields

$$
(2.12) \quad \frac{1}{\mathcal{H}_f} \left( \frac{\partial \mathcal{H}_f}{\partial p} - \frac{\partial \mathcal{H}_f}{\partial q} \right) = \int_0^1 (2t - 1)T''(t_1(t)) \, dt.
$$

The right hand side of (2.12) can be split into a sum of two integrals:

$$
\int_0^{1/2} (2t - 1)T''(t_1(t)) \, dt + \int_{1/2}^1 (2t - 1)T''(t_1(t)) \, dt.
$$

Substituting $t = 1 - v$ in first integral above yields

$$
\int_0^{1/2} (2t - 1)T''(t_1(t)) \, dt = - \int_{1/2}^1 (2v - 1)T''(t_2(v)) \, dv,
$$

where $t_2(t) = (1 - t)p + tq$. Thus (2.12) can be written as

$$
(2.13) \quad \frac{1}{\mathcal{H}_f} \left( \frac{\partial \mathcal{H}_f}{\partial p} - \frac{\partial \mathcal{H}_f}{\partial q} \right) = \int_{1/2}^1 (2t - 1) (T''(t_1(t)) - T''(t_2(t))) \, dt.
$$

By (2.4) and Mean Value Theorem, there is a $\xi$ between $|t_1(t)|$ and $|t_2(t)|$ such that

$$
T''(t_1(t)) - T''(t_2(t)) = T''(|t_1(t)|) - T''(|t_2(t)|) = (|t_1(t)| - |t_2(t)|) T''(\xi).
$$

But

$$
|t_1(t)| - |t_2(t)| = \frac{|t_1(t)|^2 - |t_2(t)|^2}{|t_1(t)| + |t_2(t)|} = \frac{(2t - 1)(p - q)(p + q)}{|t_1(t)| + |t_2(t)|}.
$$

Hence,

$$
(2.14) \quad \frac{p - q}{\mathcal{H}_f} \left( \frac{\partial \mathcal{H}_f}{\partial p} - \frac{\partial \mathcal{H}_f}{\partial q} \right) = (p - q)^2 (p + q) \int_{1/2}^1 \frac{(2t - 1)^2}{|t_1(t)| + |t_2(t)|} T''(\xi) \, dt.
$$
(2.8) shows that $\mathcal{J} < (>) 0$ is equivalent to that $T'''(t) > (>) 0$ for $t \in (0, \infty)$, and so

\[
\int_{1/2}^{1} \frac{(2t - 1)^2}{|t_1(t)| + |t_2(t)|} T'''(\xi) \, dt > (>) 0 \text{ for } p \neq \pm q.
\]

From (2.14) together with (2.15) and $\mathcal{H}_f > 0$ it follows that

\[
(p - q) \left( \frac{\partial \mathcal{H}_f}{\partial p} - \frac{\partial \mathcal{H}_f}{\partial q} \right) \left\{ \begin{array}{ll}
> 0 & \text{iff } p + q > (>) 0, \\
< 0 & \text{iff } p + q < (>) 0.
\end{array} \right.
\]

Using Theorem M-O, our required result follows. \hfill \Box

3. **An Application to the Four-parameter Homogeneous Means**

In 2005, Yang [16, 19] defined a four-parameter family generated by Stolarsky means, that is, $F(p,q;r,s;a,b) = \mathcal{H}_{F_t}(p,q;a,b)$, where $\mathcal{H}_L = \mathcal{H}_L(r,s;x,y)$, and investigated its monotonicity and log-convexity in parameters. Witkowski proved that $F(p,q;r,s;a,b)$ is a mean of positive numbers $a$ and $b$ for every $(p,q), (r,s) \in \mathbb{R}^2$ in [13, 6.4].

In the same way, Witkowski created a general four-parameter family $F_f(p,q;r,s;a,b):= \mathcal{H}_{F_f}(r,s;x,y)(p,q;a,b)$ and obtained some equivalent conditions for convexity with respect to parameter $p$ and $q$ in [13, Theorem 5.3]. But it is difficult to deal with a concrete member of the four-parameter family by using Witkowski’s results.

In this section, we will use our main result in this paper to study the Schur convexity of the four-parameter homogeneous means $F(p,q;r,s;a,b)$. We first recall the four-parameter means $F(p,q;r,s;a,b)$ as follows.

**Definition 2 ([19]).** Let $(p,q), (r,s) \in \mathbb{R}^2, (a,b) \in \mathbb{R}_+$. Then $F(p,q;r,s;a,b)$ are called four-parameter means if for $a \neq b$

\begin{align}
F(p,q;r,s;a,b) &= \left( \frac{L(a^{pr},b^{qr}) L(a^{qs},b^{qs})}{L(a^{ps},b^{ps}) L(a^{qs},b^{qs})} \right)^{\frac{1}{(p-q)(r-s)}} \text{ if } pqr s(p-q)(r-s) \neq 0, \\
\text{or} \\
F(p,q;r,s;a,b) &= \left( \frac{a^{ps} - b^{ps} a^{qs} - b^{qs}}{a^{ps} - b^{ps} a^{qs} - b^{qs}} \right)^{\frac{1}{(p-q)(r-s)}} \text{ if } pqr s(p-q)(r-s) \neq 0;
\end{align}

(3.1)
(3.2)
The four-parameter homogeneous means $\textbf{F}(p,q;r,s;a,b)$ contain many two-parameter means mentioned in [17], for example (see [18, Table 1]), $\textbf{F}(p,q;1,0;a,b)$ are exactly the Stolarsky means $S_{p,q}(a,b)$, $\textbf{F}(2,1;r,s;a,b)$ are just the Gini means $G_{p,q}(a,b)$, $\textbf{F}(1,1;r,s;a,b)$ are the two-parameter identric (exponential) means $\mathcal{H}_L(p,q;a,b)$, and $\textbf{F}(3/2,1/2;r,s;a,b)$ are the two-parameter Heronian means $\mathcal{H}_H(p,q;a,b)$, etc. It is thus clear that $\textbf{F}(p,q;r,s;a,b)$ is a class of more general means.

Concerning the Schur convexity of the four-parameter homogeneous means, we have the following

**Theorem 2.** For fixed $a, b > 0$ with $a \neq b$ and $(r,s) \in \mathbb{R}$, the four-parameter homogeneous means $\textbf{F}(p,q;r,s;a,b)$ are Schur convex if and only if $(p+q)(r+s) < 0$ and Schur concave if and only if $(p+q)(r+s) > 0$ with respect to $(p,q)$.

**Proof.** Since $\textbf{F}(p,q;r,s;a,b) = \mathcal{H}_{\mathcal{H}_L}(p,q;a,b)$, where $\mathcal{H}_L = \mathcal{H}_L(r,s;x,y) = S_{r,s}(x,y)$ is symmetric, homogenous and three-time differentiable. To prove this theorem, it is enough to verify that

$$\mathcal{J} = (x - y)(x\mathcal{I})_x < (>)0,$$

where $\mathcal{I} = (\ln \mathcal{H}_L)_{xy}$.

In fact, we have shown $\mathcal{J} < (>)0$ if and only if $r + s < (>)0$ in [19, Proof of Theorem 2.3].

By Theorem 1, our desired result follows. \[\square\]

With $(r,s) = (1,0),(2,1),((1,1),(3/2,1/2)$, we obtain immediately

**Corollary 1.** For fixed $a, b > 0$ with $a \neq b$, $S_{p,q}(a,b)$, $G_{p,q}(a,b)$, $I_{p,q}(a,b)$ and $\mathcal{H}_{p,q}(a,b)$ are all Schur convex if and only if $p + q < 0$ and Schur concave if and only if $p + q > 0$ with respect to $(p,q)$.

**Remark 4.** It is clear that Corollary 1 contains improvements for Qi’s and Sándor’s results.

**References**


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