Formulas for the Number of Spanning Trees in a Maximal Planar Map

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Abstract

The number of spanning trees of a map \( C \) denoted by \( \tau(C) \) is the total number of distinct spanning subgraphs of \( C \) that are trees. A maximal planar map is a simple graph \( G \) formed by \( n \) vertices, \( 3(n - 2) \) edges and all faces having degree 3 [2]. In this paper, we derive the explicit formula for the number of spanning trees of the maximal planar map and deduce a formula for the number of spanning trees of the crystal planar map.

Mathematics Subject Classification: 05C85, 05C30

Keywords: graphs; maps; maximal planar map; crystal planar map; fan planar map; complexity; spanning trees

1 Introduction

First of all, let’s recall some necessary definitions related to our work, an undirected graph \( G \) is a triplet \( (V_G,E_G,\delta) \) where \( V_G \) is the set of vertices of the graph \( G \), \( E_G \) is the set of edges of the graph \( G \) and \( \delta \) is the application \( \delta : E_G \rightarrow \mathcal{P}(v), e_i \mapsto \delta(e_i) = \{v_j, v_k\} \) with \( v_j \) and \( v_k \) are end vertices of the edge \( e_i \). A loop is an edge \( e_i \in E_G \) with \( v_j = v_k \), if \( \delta(e_i) = \delta(e_j) \) with \( i \neq j \) then the edges \( e_i \) and \( e_j \) are called multiple. A graph which contains neither multiple edges nor loops is called a simple graph. The degree of a vertex \( v \) noted \( \text{deg}(v) \) is the number of edges incident to it. The sum of the degrees of all vertices of a graph is equal to twice the number of its edges i.e. \( \sum_{v \in V_G} \text{deg}(v) = 2|E_G| \). A graph \( G \) is called connected if any two of its vertices may be connected by a path [12]. The graphs that we consider are in most cases connected but may contain multiple edges. A map \( C \) is a graph \( G \) drawn on a surface \( X \)
or embedded into it (that is, a compact variety 2-dimensional orientable) in such a way that: the vertices of graph are represented as distinct points of the surface, the edges are represented as curves on the surface that intersect only at the vertices, if we cut the surface along the graph thus drawn, what remains (that is, the set $X \setminus G$) is a disjoint union of connected components, called faces, each homeomorphic to an open disk [6].

A planar map is a map drawn on the plane. Through this paper, all maps are planar and connected. Euler’s formula for maps is: $|V_C| + |F_C| - |E_C| = 2 - 2g$, where $F_C$ is the set of faces of the map $C$ and $g$ is the genus of a map $C$, in the planar case $g = 0$. [6].

A tree is a connected graph without cycle. A plan tree is a tree designed in the plane. The distance between two distinct vertices $v_i$ and $v_j$ of a tree, denoted by $d(v_i, v_j)$ is equal to the length of the shortest path (number of edges in) that connects $v_i$ and $v_j$, by Conventionally, $d(v_i, v_i) = 0$. We define a complete vertex in a graph $G$ by the vertex $v_0$ such that $d(u, v_0) = 1$ for each $u \in V_G \setminus \{v_0\}$. In a complete graph all the vertices are completes.

The Wiener index of a connected graph is the sum of distances between all pairs of vertices [3], [4], the Wiener index of a connected graph $G$ is defined as:

$$W(G) = \sum_{\{v_i, v_j\} \subseteq V(G)} d(v_i, v_j).$$

The particular case of trees has been echoed by several people. For a tree with $n$ vertices, the Wiener index $W(T_n)$ is minimized by the star tree with $n$ vertices and maximized by the path with $n$ vertices [12]. In [3] we worked on the Wiener index in the case of planar maps and gave an inequality, which minimizes and maximizes any map by the maximal planar map with $n$ vertices and the path of $n$ vertices and also we gave a formula which calculates the Wiener index of maximal planar map $E_n$, our results given in the following theorem:

**Theorem 1.** Let $C_n$ be a map with $n$ vertices, then

$$(n - 2)^2 + 2 = W(E_n) \leq W(C_n) \leq W(P_n) = \frac{n(n^2 - 1)}{6}.$$
define the $n \times n$ characteristic matrix $L = [a_{ij}]$ as follows: (i) $a_{ij} = -p(v_i, v_j)$ if $i \neq j$, where $p(v_i, v_j)$ is the number of edges that connects $v_i$ with $v_j$, (ii) $a_{ij}$ equals the degree of vertex $v_i$ if $i = j$, and (iii) $a_{ij} = 0$ otherwise. The Kirchhoff matrix tree theorem states that all cofactors of $L$ are equal, and their common value is $\tau(C)$. The matrix tree theorem can be applied to any map $C$ to determine $\tau(C)$, but this requires evaluating a determinant of a corresponding characteristic matrix. However, for a few special families of graphs there exist simple formulas which make it much easier to calculate and determine the number of corresponding spanning trees especially when these numbers are very large. One of the first such results is due to Cayley who showed that complete graph on $n$ vertices, $K_n$, has $n^{n-2}$ spanning trees [1], that is he showed

$$\tau(K_n) = n^{n-2} \quad \text{for} \quad n \geq 2.$$  

**Theorem 2.** [12] If $L^*(C)$ is a matrix obtained by deleting row $i$ and column $j$ of the Laplacian matrix $L(C)$, then $\tau(C) = (-1)^{i+j} \det L^*(C)$. 

Another result is due to Seldacek [10] who derived a formula for the wheel on $n + 1$ vertices, $W_{n+1}$, which is formed from a cycle $C_n$ on $n$ vertices by adding a vertex adjacent to every vertex of $C_n$ (see Fig. 2). In particular, he showed that

$$\tau(W_{n+1}) = \left(\frac{3 + \sqrt{5}}{2}\right)^n + \left(\frac{3 - \sqrt{5}}{2}\right)^n - 2 \quad \text{for} \quad n \geq 3.$$  

In [8], we gave a formula to calculate the complexity of the $n$-Grid chains $G_n$ where $n$ is the number of squares ($|V_{G_n}| = 2n + 2$) (see Fig. 2) is as follows:

$$\tau(G_n) = \frac{1}{2\sqrt{3}} \left( (2 + \sqrt{3})^{n+1} - (2 - \sqrt{3})^{n+1} \right), \quad n \geq 1,$$
also another formula to calculate the complexity of the n-Fan chains $\mathcal{F}_n$ where $n$ is the number of triangles ($|V_{\mathcal{F}_n}| = n + 2$) (see Fig. 2) is as follows:

$$\tau(\mathcal{F}_n) = \frac{1}{\sqrt{5}} \left( \left( \frac{3 + \sqrt{5}}{2} \right)^{n+1} - \left( \frac{3 - \sqrt{5}}{2} \right)^{n+1} \right), \quad n \geq 1.$$ 

Figure 2: The Wheel, the n-Grid chains and the n-Fan chains

In addition, other formulas are related to our work can also be found in [7], [10], [11]. In this paper, we derive the explicit formula for the number of spanning trees of a maximal planar map and deduce a formula for the number of spanning trees of the crystal planar map.

\section{Main Results}

Before presenting the main results, we need the following lemmas:

**Lemma 1.** Let $\mathcal{C}$ be a map and $\tau(\mathcal{C})$ denote the number of spanning trees of $\mathcal{C}$. If $e = v_iv_{i+1} \in E(\mathcal{C})$ ($e$ is not a loop), then

$$\tau(\mathcal{C}) = \tau(\mathcal{C} - e) + \tau(\mathcal{C}.e).$$ 

**Proof:** See [8], [12].

Let $\mathcal{C}$ be a map of type $\mathcal{C} = C_1 \mid C_2$, $\mathcal{C}$ is a map such that $v_1$ and $v_2$ two vertices of $\mathcal{C}$ connected by an edge $e$ [2](see Fig. 3).

Figure 3: A map $\mathcal{C} = C_1 \mid C_2$
Property 1. Let \( C \) be a map of type \( C = C_1 \mid C_2 \)
- \( C_1 \) and \( C_2 \) have two common vertices \( v_1, v_2 \), a common edge \( e \) and a common face (the external face).
- \( V_C = V_{C_1} + V_{C_2} - 2, E_C = E_{C_1} + E_{C_2} - 1 \) and \( F_C = F_{C_1} + F_{C_2} - 1 \).

Example 1. Here is an example of maps \( C, C_1, C_2, C_1 - e, C_2 - e, C_1.e \) and \( C_2.e \) (see Fig. 4).

\[
\begin{align*}
\text{c} & \quad \rightarrow \quad \text{c}_1 \quad \text{c}_2 \\
\text{c}_1 - e & \quad \text{c}_2 - e \\
\text{c}_1.e & \quad \text{c}_2.e
\end{align*}
\]

Figure 4: An example of maps \( C, C_1, C_2, C_1 - e, C_2 - e, C_1.e \) and \( C_2.e \)

Lemma 2. Let \( C \) be a map of type \( C = C_1 \mid C_2 \) (see Fig. 3), then
\[
\tau(C) = \tau(C_1) \times \tau(C_2) - \tau(C_1 - e) \times \tau(C_2 - e).
\]

Proof: See [8].

Definition 1. (A maximal planar map [2]) Let \( E_n \) be a family of maps that contains:
- \( n \) vertices, two complete vertices of degree \( n - 1 \), two vertices of degree 3 and \( n - 4 \) vertices of degree 4,
- \( 2(n - 2) \) faces of degree 3 (all faces having degree 3),
- \( 3(n - 2) \) edges,

then the family of this maps is called a maximal planar map.

The maps \( E_3 \) and \( E_4 \) are presented in the example 3. For \( E_5, E_6, \) and \( E_n \) (see Fig. 5)

Example 2. In the maps \( E_3 \) and \( E_4 \), all the vertices are complete. In other words, we say that the map is complete (see Fig 6).
Lemma 3. Let $\mathcal{E}_n$ be a maximal planar map with $n$ vertices, then

$$n/\tau(\mathcal{E}_n), \quad n \geq 3.$$ 

Proof: We first form the Kirchhoff characteristic matrix $L_n$ ($n \times n$) corresponding to the labeling of $\mathcal{E}_n$ shown as follows (in Figure 7):

$$L_n(\mathcal{E}_n) = 
\begin{bmatrix}
    v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & \ldots & v_{n-2} & v_{n-1} & v_n \\
    n-1 & -1 & -1 & -1 & -1 & -1 & \ldots & -1 & -1 & -1 \\
    -1 & n-1 & -1 & -1 & -1 & -1 & \ldots & -1 & -1 & -1 \\
    -1 & -1 & 3 & -1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
    -1 & -1 & -1 & 1 & 4 & 1 & 0 & \ldots & 0 & 0 & 0 \\
    -1 & -1 & 0 & -1 & 4 & -1 & 0 & \ldots & 0 & 0 & 0 \\
    -1 & -1 & 0 & 0 & -1 & 4 & 0 & \ldots & 0 & 0 & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    -1 & -1 & 0 & 0 & 0 & 0 & \ldots & 4 & -1 & 0 \\
    -1 & -1 & 0 & 0 & 0 & 0 & \ldots & -1 & 4 & -1 \\
    -1 & -1 & 0 & 0 & 0 & 0 & \ldots & 0 & -1 & 3
\end{bmatrix}.$$ 

Figure 7: The principal matrix of $\mathcal{E}_n$

Next, we focus our attention on the principal submatrix $L_{n-1}$ which obtained by canceling its last row and column corresponding to vertex $v_n$. So, the number of spanning trees of a maximal planar map $\mathcal{E}_n$ equals $\tau(\mathcal{E}_n) = det(L_{n-1})$. 
we denote \( r_i \) by the \( i \)-th row and \( c_i \) by the \( i \)-th column of the determinant. In the previous determinant, we replace \( c_1 \) by \( c_1 + c_2 + \ldots + c_{n-1} \), i.e., we add to the first column the sum of other (transformation is symbolized as follows: \( c_1 \leftarrow \sum_{i=1}^{n-1} c_i \), i.e., this does not change the determinant, then we obtain:

\[
\tau(E_n) = \begin{vmatrix}
1 & -1 & -1 & -1 & -1 & \ldots & -1 & -1 \\
0 & 1 & -1 & -1 & -1 & \ldots & -1 & -1 \\
0 & 1 & -1 & -1 & -1 & \ldots & -1 & -1 \\
0 & 1 & -1 & -1 & -1 & \ldots & -1 & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & -1 & 0 & 0 & 0 & \ldots & 4 & -1 \\
-1 & -1 & 0 & 0 & 0 & \ldots & -1 & 4
\end{vmatrix}
\]

Next, we replace \( c_j \) by \( c_1 + c_j \) for \( j = 2, \ldots, n-1 \), i.e., \( c_j \leftarrow c_1 + c_j \), we obtain:

\[
\tau(E_n) = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
1 & n & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & -1 & 3 & -1 & 0 & \ldots & 0 & 0 \\
0 & -1 & -1 & 4 & -1 & \ldots & 0 & 0 \\
0 & -1 & -1 & 4 & -1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & -1 & 0 & 0 & 0 & \ldots & 4 & -1 \\
1 & -1 & 0 & 0 & 0 & \ldots & -1 & 4
\end{vmatrix}
\]

Expanding \( L_{n-1} \) along the first row we obtain the determinant of order \((n - 2) \times (n - 2)\) and expanding the determinant obtained along the first row we obtain the determinant of order \((n - 3) \times (n - 3)\) as follows:
Lemma 4. Let $\mathcal{E}_n$ be a maximal planar map with $n$ vertices, then

$$\frac{\tau(\mathcal{E}_n)}{n} = \frac{4\tau(\mathcal{E}_{n-1})}{n-1} - \frac{\tau(\mathcal{E}_{n-2})}{n-2}, \quad n \geq 5.$$ 

Proof: By lemma 3, we have the determinant of order $(n-3) \times (n-3)$:

$$\tau(\mathcal{E}_n) = \begin{vmatrix} 3 & -1 & 0 & 0 & \ldots & 0 & 0 \\ -1 & 4 & -1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 4 & -1 & \ldots & 0 & 0 \\ 0 & 0 & -1 & 4 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 4 & -1 \\ 1 & 1 & 1 & 1 & \ldots & 0 & 5 \end{vmatrix}$$

In the previous determinant, we denote by $c_i$ by the $i$-th column and $r_i$ by the $i$-th row of the determinant, $c_i \leftarrow c_i - c_{i+1}$ for $i = 1, \ldots, n-4$, we obtain the determinant as follows:

$$\tau(\mathcal{E}_n) = \begin{vmatrix} 4 & -1 & 0 & 0 & \ldots & 0 & 0 \\ -5 & 5 & -1 & 0 & \ldots & 0 & 0 \\ 1 & -5 & 5 & -1 & \ldots & 0 & 0 \\ 0 & 1 & -5 & 5 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 5 & -1 \\ 0 & 0 & 0 & 0 & \ldots & -5 & 5 \end{vmatrix}$$

In the end, $r_i \leftarrow \sum_{j=1}^{i} r_j$ for $i = 2, \ldots, n-3$, we obtain the determinant as follows:
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\[ \tau(E_n) = \begin{vmatrix} 4 & -1 & 0 & 0 & \ldots & 0 & 0 \\ -1 & 4 & -1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 4 & -1 & \ldots & 0 & 0 \\ 0 & 0 & -1 & 4 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 4 & -1 \\ 0 & 0 & 0 & 0 & \ldots & -1 & 4 \end{vmatrix} \]

then \( \tau(E_n) = 4 \Delta_{n-1} - \Delta_{n-2} \), where \( \Delta_i = \text{det}(D_i) \) (\( \Delta_i = 4 \Delta_{i-1} - \Delta_{i-2} \) because \( \Delta_i \) is tri-diagonal matrix), and \( D_i \) is defined as following:

\[ D_i = \begin{bmatrix} 4 & -1 & & & & \\ -1 & 4 & -1 & & & \\ & -1 & 4 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 4 & -1 \\ & & & & -1 & 4 \end{bmatrix}_{i \times i} \]

With the same technique, we obtain: \( \frac{\tau(E_{n-1})}{n-1} = \Delta_{n-1} \) and \( \frac{\tau(E_{n-2})}{n-2} = \Delta_{n-2} \), hence the result. \( \square \)

3 Applications

Theorem 3. (A maximal planar map) The complexity of the maximal planar map \( E_n \) (see Fig. 5) is given by the following formula:

\[ \tau(E_n) = \frac{n}{2\sqrt{3}} \left( (2 + \sqrt{3})^{n-2} - (2 - \sqrt{3})^{n-2} \right), \quad n \geq 3. \]

Proof: \( \tau(E_3) = 3, \tau(E_4) = 16, \tau(E_5) = 75, \)

\[ \tau(E_n) = \frac{4\tau(E_{n-1})}{n-1} - \frac{\tau(E_{n-2})}{n-2} \] (by lemma 4), hence we obtain the system:

\[
\left\{ \begin{array}{l}
\frac{\tau(E_n)}{n} = \frac{4\tau(E_{n-1})}{n-1} - \frac{\tau(E_{n-2})}{n-2}, \\
\tau(E_3) = 3 \\
\tau(E_4) = 16
\end{array} \right.
\]

Then we get a sequence such that \( u_n = 4u_{n-1} - u_{n-2}, n \geq 3 \), therefore the characteristic equation is \( r^2 - 4r + 1 = 0 \), so the solutions of this equation are:
Let $F_n$ and $G_n$ be the maps as follows (see Figure 8):

![Figure 8: The maps $F_n$ and $G_n$](image)

**Theorem 4. (A Fan planar map)** The complexity of the Fan planar maps $F_n$ and $G_n$ (see Fig. 8) is given by the following formulas:

\[
\tau(F_n) = \frac{1}{\sqrt{3}} \left( (2 + \sqrt{3})^{n-1} - (2 - \sqrt{3})^{n-1} \right), \quad n \geq 2,
\]

\[
\tau(G_n) = \frac{1}{2} (\sqrt{3} - 1) \left( (2 + \sqrt{3})^{n-1} - (2 - \sqrt{3})^{n-2} \right), \quad n \geq 2.
\]

**Proof:** We put $f_n = \tau(F_n)$ and $g_n = \tau(G_n)$. $f_2 = 2, g_2 = 1$, in the map $F_n$, we cut the first cycle (see Fig. 8), and we use Lemma 2 (the same goes for the map $G_n$), then we obtain: $\tau(G_n') = 3\tau(F_{n-1}) - \tau(G_{n-1})$, $\tau(F_n) = 2\tau(G_n) - \tau(F_{n-1})$ therefore, we have the following system:

\[
\begin{align*}
&f_n = 2g_n - f_{n-1} \\
g_n = 3f_{n-1} - g_{n-1}
\end{align*}
\] with $f_2 = 2$ and $g_2 = 1$,
we replace by the value of \( g_n \) in the first equation, we get:

\[
\begin{align*}
\begin{cases}
  f_n &= 5f_{n-1} - 2g_{n-1} \\
  g_n &= 3f_{n-1} - g_{n-1}
\end{cases}
\quad \text{with } f_2 = 2 \text{ and } g_2 = 1
\end{align*}
\]

\[
\begin{pmatrix} f_n \\ g_n \end{pmatrix} = M \begin{pmatrix} f_{n-1} \\ g_{n-1} \end{pmatrix}, \quad \text{where } M = \begin{pmatrix} 5 & -2 \\ 3 & -1 \end{pmatrix},
\]

\[
\begin{pmatrix} f_n \\ g_n \end{pmatrix} = M^{n-2} \begin{pmatrix} f_2 \\ g_2 \end{pmatrix},
\]

then, we compute \( M^{n-2} \):

\[
\det (M - \lambda I_2) = \lambda^2 - 4\lambda + 1 = 0, \quad \lambda_1 = 2 - \sqrt{3} \text{ and } \lambda_2 = 2 + \sqrt{3}, \quad \lambda_1 \neq \lambda_2
\]

then there is \( P \) invertible such that \( M = PDP^{-1} \) where

\[
D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},
\]

\( P \) is the transformation matrix formed by eigenvectors

\[
P = \begin{pmatrix} \frac{1}{3+\sqrt{3}} & \frac{1}{3-\sqrt{3}} \\ \frac{3+\sqrt{3}}{2} & \frac{3-\sqrt{3}}{2} \end{pmatrix},
\]

\[
P^{-1} = \frac{-1}{\sqrt{3}} \begin{pmatrix} \frac{3-\sqrt{3}}{2} & -1 \\ \frac{3+\sqrt{3}}{2} & 1 \end{pmatrix},
\]

\( M^{n-2} = PD^{n-2}P^{-1} \) where

\[
D^{n-2} = \begin{pmatrix} (2 - \sqrt{3})^{n-2} & 0 \\ 0 & (2 + \sqrt{3})^{n-2} \end{pmatrix},
\]

from which we obtain \( M^{n-2} \), then

\[
\begin{pmatrix} f_n \\ g_n \end{pmatrix} = M^{n-2} \begin{pmatrix} f_2 \\ g_2 \end{pmatrix},
\]

hence the result. \( \square \)

Let \( \mathcal{E}_n \) be a maximal planar map shown in Fig. 5. If we delete the edge \( e = v_1v_2 \) of \( \mathcal{E}_n \), we obtain the crystal planar map \( \mathcal{C}_n \) with \( n \) vertices (see Fig. 9).
Theorem 5. (A crystal planar map) The complexity of the crystal planar map $C_n$ (see Fig. 9) is given by the following formula:

$$
\tau(C_n) = \frac{1}{2\sqrt{3}}(n-2)\left((2+\sqrt{3})^{n-2} - (2-\sqrt{3})^{n-2}\right), \quad n \geq 3.
$$

Proof: By Lemma 1, we have: $\tau(E_n) = \tau(E_n - e) + \tau(E_n.e)$, since $\tau(E_n - e) = \tau(C_n)$ and $\tau(E_n.e) = \tau(F_n)$ (see Fig. 10):

then $\tau(C_n) = \tau(E_n) - \tau(F_n)$. We replace by the value of $\tau(E_n)$ from Theorem 3 and the value of $\tau(F_{n-1})$ from Theorem 4, thus our result follows. □

4 Conclusion

In this paper, we have interested by calculating the number of spanning trees of some particular planar maps, for that we have derived the explicit formula to calculate the number of spanning trees of the maximal planar map, then we have deduced the formula for the number of spanning trees of the crystal planar map.
References


Received: April, 2011