Weak Convergence Theorem for Finding Common Fixed Points of a Family of Firmly Nonexpansive Mappings and a Nonspraying Mapping in Hilbert Spaces

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Abstract

In this paper, we introduce an iterative method and prove a weak convergence theorem for finding common fixed points of a family of firmly nonexpansive mappings and a nonspraying mapping in Hilbert spaces. Moreover, we apply our result to finding common element of a solution set of generalized mixed equilibrium problem and a common fixed point set nonspraying mappings. Using the result, we improve and unify several results in fixed point problems and equilibrium problems.

Keywords: Equilibrium problem, Fixed point problem, Firmly nonexpansive mapping, Nonspraying mapping

1 Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Then a mapping $T : C \rightarrow C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of $T$. A mapping $F$ is said to be firmly nonexpansive if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy \rangle,$$

for all $x, y \in C$; see, for instance, [2, 4, 5, 13, 14]. On the other hand, a mapping $Q : C \rightarrow C$ is said to be quasi-nonexpansive if $F(Q) \neq \emptyset$ and

$$\|Qx - y\| \leq \|x - y\|,$$

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for all $x \in C$ and $y \in F(Q)$, where $F(Q)$ is the set of fixed points of $Q$. If $T : C \rightarrow C$ is nonexpansive and the set $F(T)$ of fixed points of $T$ is nonempty, then $T$ is quasi-nonexpansive.

Recently, Kohsaka and Takahashi [8] introduced the following nonlinear mapping: Let $E$ be a a Hilbert space and let $C$ be a nonempty closed convex subset of $E$. Then, a mapping $S : C \rightarrow C$ is said to be nonspraying if

$$2\|Sx - Sy\|^2 \leq \|Sx - y\|^2 + \|x - Sy\|^2,$$

for all $x, y \in C$. We know in a Hilbert space that every firmly nonexpansive mapping is nonspraying and that if the set of fixed points of a nonspraying mapping is nonempty, the nonspraying mapping is quasi-nonexpansive; see [8]. Let $A : C \rightarrow H$ be a mapping of $C$ into $H$ is called monotone if

$$\langle Au - Av, u - v \rangle \geq 0 \ \forall u, v \in C.$$

A mapping $A : C \rightarrow H$ is called $\lambda$-inverse-strongly monotone if there exists a positive real number $\lambda$ such that

$$\langle Au - Av, u - v \rangle \geq \lambda \|Ax - Ay\|^2 \ \forall u, v \in C.$$

Let $F : C \times C \rightarrow \mathbb{R}$ be a bifunction. The equilibrium problem for $F$ is to determine its equilibrium points, i.e. the set

$$EP(F) = \{x \in C : F(x, y) \geq 0, \ \forall y \in C\}.$$

Many problems in physics, optimization, and economics require some elements of $EP(F)$, see [2, 3, 9, 15, 16, 17]. Several iterative methods have been proposed to solve the equilibrium problem, see for instance [3, 15, 16, 17]. In 2005, Combettes and Hirstoaga [3] introduced an iterative scheme for finding the best approximation to the initial data when $EP(F)$ is nonempty and proved a strong convergence theorem.

The variational inequality problem is to find $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0$$

for all $v \in C$. The set of solutions of the variational inequality is denoted by $VI(C, A)$. The generalized equilibrium problem for $F$ and $A$ is to find $x \in C$ such that

$$F(x, y) + \langle Ax, y - x \rangle \geq 0 \ \text{for all} \ y \in C. \quad (1.1)$$

Problem (1.1) was introduce by Takahashi and Takahashi [16] and the set of solution of (1.1) is denoted by $GEP(F, A)$. The generalized mixed equilibrium problem for $F, \psi$ and $A$ is to find $x \in C$ such that

$$F(x, y) + \psi(y) - \psi(x) + \langle Ax, y - x \rangle \geq 0 \ \text{for all} \ y \in C. \quad (1.2)$$
The set of solution of (1.2) is denoted by $GMEP(F, \varphi, A)$.

On the other hand, Halpern [6] introduced the following iterative scheme for approximating a fixed point of $T$:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n$$

(1.3)

for all $n \in \mathbb{N}$, where $x_1 = x \in C$ and $\{\alpha_n\}$ is a sequence of $[0, 1]$. Recently, Aoyama et al. [1] introduced a Halpern type iterative sequence for finding a common fixed point of a countable family of nonexpansive mappings. Let $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)T_n x_n$$

(1.4)

for all $n \in \mathbb{N}$, where $C$ is a nonempty closed convex subset of a Banach space, $\{\alpha_n\}$ is a sequence in $[0, 1]$ and $\{T_n\}$ is a sequence of nonexpansive mappings of $C$ into itself which satisfies the AKTT-condition, that is,

$$\sum_{n=1}^{\infty} \sup \{\|T_{n+1}z - T_nz\| : z \in C\} < \infty.$$  

(1.5)

They proved that the sequence $\{x_n\}$ defined by (1.4) converges strongly to a common fixed point of $\{T_n\}$.

In this paper, motivated by Plubtieng and Thammathiwat [12], Iemoto and Takahashi [7], we introduce a new iterative sequence and prove a weak convergence theorem for finding common fixed points of a families of nonexpansive mappings and a nonspreading mapping in Hilbert spaces. Moreover, we apply our result to finding common element of a solution set of generalized mixed equilibrium problem and a common fixed point set nonspreading mappings.

2 Preliminaries

This section collects some lemmas which will be used in the proofs for the main results in the next section. Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. In a Hilbert space, it is known that

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2.$$

By definition of the metric projection $P_C$ we known that $P_C$ is a nonexpansive mapping of $H$ onto $C$ and satisfies

$$\|P_Cx - P_Cy\|^2 \leq \langle P_Cx - P_Cy, x - y \rangle, \ \forall x, y \in H.$$

Further, for any $x \in H$ and $y \in C$, $y = P_Cx$ if and only if $\langle x - y, y - z \rangle \geq 0$, $\forall z \in C$. 


A space $X$ is said to satisfy Opial's condition [10] if for each sequence \( \{x_n\}_{n=1}^{\infty} \) in $X$ which converges weakly to point $x \in X$, we have
\[
\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|, \ \forall y \in X, \ y \neq x
\]
and
\[
\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|, \ \forall y \in X, \ y \neq x.
\]

**Lemma 2.1.** [8] Let $H$ be a Hilbert space, $C$ a nonempty closed convex subset of $H$. Let $S$ be a nonspreading mapping of $C$ into itself. Then $F(S)$ is closed and convex.

In order to prove the main result, we shall use the following lemmas in the sequel.

**Lemma 2.2.** [7] Let $H$ be a Hilbert space, $C$ a closed convex subset of $H$, and $S : C \to C$ a nonspreading mapping with $F(S) \neq \emptyset$. Then $S$ is demiclosed, i.e., $x_n \rightharpoonup u$ and $x_n - Sx_n \to 0$ imply $u \in F(S)$.

**Lemma 2.3.** [7] Let $H$ be a Hilbert space, $C$ a nonempty closed convex subset of a real Hilbert space $H$ and let $S$ be a nonspreading mapping of $C$ into itself and let $A = I - S$. Then
\[
\|Ax - Ay\|^2 \leq \langle x - y, Ax - Ay \rangle + \frac{1}{2}(\|Ax\|^2 + \|Ay\|^2).
\]

**Lemma 2.4.** [11] Let $C$ be a nonempty bounded closed convex subset of Hilbert space $E$ and $\{T_n\}$ a sequence of mappings of $C$ into itself. Suppose that
\[
\lim_{k,l \to \infty} \rho_k^l = 0
\]
where $\rho_k^l = \sup\{\|T_kz - T_lz\| : z \in C\} < \infty$, for all $k, l \in \mathbb{N}$. Then for each $x \in C$, $\{T_nx\}$ converges strongly to some point of $C$. Moreover, let $T$ be a mapping from $C$ in to itself defined by
\[
Tx = \lim_{n \to \infty} T_nx, \ \text{for all } x \in C.
\]
Then $\lim_{n \to \infty} \sup\{\|Tz - T_nz\| : z \in C\} = 0$.

In fact, Aoyama et al. [1] proved Lemma 2.4 in case the sequence $\{T_n\}$ satisfies the AKTT-condition. We note that if a sequence $\{T_n\}$ satisfies the AKTT-condition then $\{T_n\}$ satisfies the condition (2.1).
3 Weak convergence theorem

In this section, we prove a weak convergence theorem for finding common fixed points of a family of nonexpansive mappings and a nonspreading mapping in Hilbert space.

**Theorem 3.1.** Let $H$ be a real Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $S$ be a nonspreading mapping of $C$ into itself and let $\{T_n\}$ be the sequences of firmly nonexpansive mappings of $C$ into itself such that $F(S) \cap (\cap_{n=1}^{\infty} F(T_n))$ is nonempty. Let $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$. Let $\{x_n\}$ be a sequence defined by $x_0 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) ST_n x_n, \quad n \geq 0. \quad (3.1)$$

Suppose that $\{T_n\}$ satisfy the AKTT-condition and $T$ be the mappings of $C$ into itself defined by $Ty = \lim_{n \to \infty} T_n y$ for all $y \in C$ and suppose that $F(T) = \cap_{n=1}^{\infty} F(T_n)$. Then $\{x_n\}$ converges weakly to $\hat{x} \in F(S) \cap (\cap_{n=1}^{\infty} F(T_n))$.

**Proof.** Take a point $v \in F(S) \cap (\cap_{n=1}^{\infty} F(T_n))$ and put $y_n = T_n x_n$. We shall show that the sequences $\{x_n\}$ is bounded. First, we note that

$$\|Sy_n - v\| \leq \|y_n - v\| = \|T_n x_n - v\| \leq \|x_n - v\|,$$

we obtain,

$$\|x_{n+1} - v\|^2 = \|\alpha_n x_n + (1 - \alpha_n) Sy_n - v\|^2$$

$$= \|\alpha_n (x_n - v) + (1 - \alpha_n) (Sy_n - v)\|^2$$

$$= \alpha_n \|x_n - v\|^2 + (1 - \alpha_n) \|Sy_n - v\|^2 - \alpha_n (1 - \alpha_n) \|Sy_n - x_n\|^2$$

$$\leq \alpha_n \|x_n - v\|^2 + (1 - \alpha_n) \|y_n - v\|^2 - \alpha_n (1 - \alpha_n) \|Sy_n - x_n\|^2$$

$$\leq \alpha_n \|x_n - v\|^2 + (1 - \alpha_n) \|x_n - v\|^2 - \alpha_n (1 - \alpha_n) \|Sy_n - x_n\|^2$$

$$= \|x_n - v\|^2 - \alpha_n (1 - \alpha_n) \|Sy_n - x_n\|^2$$

$$\leq \|x_n - v\|^2.$$

Hence $\{\|x_{n+1} - v\|\}$ is a decreasing sequence and therefore $\lim_{n \to \infty} \|x_n - v\|$ exists. This implies that $\{x_n\}, \{y_n\}$ and $\{Sy_n\}$ are bounded. Since $\{T_n\}$ is firmly nonexpansive, it follows that

$$\|T_n x_n - v\|^2 = \|T_n x_n - T_n v\|^2$$

$$\leq \langle T_n x_n - v, x_n - v \rangle$$

$$= \frac{1}{2} (\|T_n x_n - v\|^2 + \|x_n - v\|^2 - \|x_n - T_n x_n\|^2),$$
for all \( v \in F(S) \cap (\cap_{n=1}^{\infty} F(T_n)) \) and hence
\[
\|T_nx_n - v\|^2 \leq \|x_n - v\|^2 - \|x_n - T_nx_n\|^2.
\]
Thus, we have
\[
\|x_{n+1} - v\|^2 = \|x_n + (1 - \alpha_n)Sy_n - v\|^2 \\
= \|\alpha_n(x_n - v) + (1 - \alpha_n)(Sy_n - v)\|^2 \\
\leq \alpha_n\|x_n - v\|^2 + (1 - \alpha_n)\|Sy_n - v\|^2 \\
\leq \alpha_n\|x_n - v\|^2 + (1 - \alpha_n)\|v - y_n\|^2 \\
= \alpha_n\|x_n - v\|^2 + (1 - \alpha_n)\|T_nx_n - v\|^2 \\
\leq \alpha_n\|x_n - v\|^2 + (1 - \alpha_n)\left(\|x_n - v\|^2 - \|x_n - T_nx_n\|^2\right).
\]
we obtain
\[
(1 - \alpha_n)\|x_n - T_nx_n\|^2 \leq \alpha_n\|x_n - v\|^2 + (1 - \alpha_n)\|x_n - v\|^2 - \|x_{n+1} - v\|^2 \\
= \|x_n - v\|^2 - \|x_{n+1} - v\|^2.
\]
Since, \( 0 < a \leq \alpha_n \leq b < 1 \) and \( \lim_{n \to \infty} \|x_n - v\|^2 = \lim_{n \to \infty} \|x_{n+1} - v\|^2 \), we obtain
\[
\|x_n - T_nx_n\| = \|x_n - y_n\| \to 0.
\]
Put \( A_n = I - ST_n \). From \( A_nv = 0 \), it follows by Lemma 2.3 that
\[
\|x_{n+1} - v\|^2 = \|\alpha_nx_n + (1 - \alpha_n)ST_nx_n - v\|^2 \\
= \|\alpha_nx_n + (1 - \alpha_n)x_n - (1 - \alpha_n)x_n + (1 - \alpha_n)ST_nx_n - v\|^2 \\
= \|x_n - v - (1 - \alpha_n)(x_n - ST_nx_n)\|^2 \\
= \|x_n - v\|^2 - 2(1 - \alpha_n)x_n - v, A_nx_n) + (1 - \alpha_n)^2\|A_nx_n\|^2 \\
= \|x_n - v\|^2 - 2(1 - \alpha_n)x_n - v, A_nx_n - A_nv) + (1 - \alpha_n)^2\|A_nx_n\|^2 \\
\leq \|x_n - v\|^2 - 2(1 - \alpha_n)x_n - v, A_nx_n - A_nv) + (1 - \alpha_n)^2\|A_nx_n\|^2 \\
+ (1 - \alpha_n)^2\|A_nx_n\|^2 \\
= \|x_n - v\|^2 - (1 - \alpha_n)\|A_nx_n\|^2 + (1 - \alpha_n)^2\|A_nx_n\|^2 \\
= \|x_n - v\|^2 - \alpha_n(1 - \alpha_n)\|A_nx_n\|^2
\]
and hence
\[
\alpha_n(1 - \alpha_n)\|A_nx_n\|^2 \leq \|x_n - v\|^2 - \|x_{n+1} - v\|^2.
\]
Since \( \lim \inf_{n \to \infty} \alpha_n(1 - \alpha_n) > 0 \), we get
\[
\lim_{n \to \infty} \|A_nx_n\| = \lim_{n \to \infty} \|x_n - ST_nx_n\| = 0.
\]
So, we have
\[
\|y_n - S y_n\| = |T_n x_n - S T_n x_n| = |T_n x_n - x_n + x_n - S T_n x_n| \\
\leq \|T_n x_n - x_n\| + \|x_n - S T_n x_n\| \to 0 \text{ as } n \to \infty.
\]

Since \(\{y_n\}\) is bounded, there exists a subsequence \(\{y_{n_i}\}\) of \(\{y_n\}\) which converges weakly to \(\hat{z}\). Without loss of generality, we can assume that \(y_{n_i} \rightharpoonup \hat{z}\). By Lemma 2.2, we have \(\hat{z} \in F(S)\). From \(\lim_{n \to \infty} \|x_n - y_n\| \to 0\) and \(y_{n_i} \rightharpoonup \hat{z}\), we get \(x_{n_i} \rightharpoonup \hat{z}\). We shall show that \(\hat{z} \in F(T)\). From \(\|T_n x_n - x_n\| \to 0\) and AKTT-condition, we have \(\|T x_n - x_n\| \leq \|T x_n - T_n x_n\| + \|T_n x_n - x_n\| \to 0\).

We next show that \(\hat{z} \in F(T)\). Assume \(\hat{z} \notin F(T)\). Since \(x_{n_i} \rightharpoonup \hat{z}\) and \(\hat{z} \neq T \hat{z}\). By the Opial’s condition, we have
\[
\liminf_{n \to \infty} \|x_{n_i} - \hat{z}\| < \liminf_{n \to \infty} \|x_{n_i} - T \hat{z}\| \\
\leq \liminf_{n \to \infty} \{\|x_{n_i} - T x_{n_i}\| + \|T x_{n_i} - T \hat{z}\|\} \\
\leq \liminf_{n \to \infty} \|x_{n_i} - \hat{z}\|.
\]

This is a contradiction. So, we get \(\hat{z} \in F(T)\). Hence \(\hat{z} \in F(S) \cap (\cap_{n=1}^{\infty} F(T_n))\).

Let \(\{x_{n_k}\}\) be another subsequence of \(\{x_n\}\) such that \(\{x_{n_k}\}\) converges weakly to \(\hat{z}\). We may show that \(\hat{z} = \tilde{z}\), suppose not. Since \(\lim_{n \to \infty} \|x_n - v\|\) exists for all \(v \in F(S) \cap (\cap_{n=1}^{\infty} F(T_n))\), it follows by the Opial’s condition that
\[
\lim_{n \to \infty} \|x_n - \hat{z}\| = \liminf_{i \to \infty} \|x_{n_i} - \tilde{z}\| < \liminf_{i \to \infty} \|x_{n_i} - \tilde{z}\| = \lim_{n \to \infty} \|x_n - \tilde{z}\| \\
= \liminf_{k \to \infty} \|x_{n_k} - \hat{z}\| < \liminf_{k \to \infty} \|x_{n_k} - \hat{z}\| = \lim_{n \to \infty} \|x_n - \hat{z}\|.
\]

This is a contradiction. Thus, we have \(\hat{z} = \tilde{z}\). This implies that \(\{x_n\}\) converges weakly to \(\hat{z} \in F(S) \cap (\cap_{n=1}^{\infty} F(T_n))\). This completes the proof. \(\square\)

4 Applications

In this section, using Theorem 3.1, we prove weak convergence theorem for finding a common element of the set of solutions of generalized mixed equilibrium problem and the fixed point set of a nonspreading mapping in Hilbert space. Before, proving our theorems, we need the following lemmas. For solving the equilibrium problem for a bifunction \(F : C \times C \to \mathbb{R}\), let us assume that \(F\) satisfies following conditions:

(A1) \(F(x, x) = 0\) for all \(x \in C\).

(A2) \(F\) is monotone, that is, \(F(x, y) + F(y, x) \leq 0\) for all \(x, y \in C\).
(A3) for each \( x, y, z \in C \). \( \lim_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y) \).

(A4) for each \( x \in C, y \mapsto F(x, y) \) is convex and lower semicontinuous.

The following lemma appears implicitly in [2].

**Lemma 4.1.** [2] Let \( C \) be a nonempty closed convex subset of \( H \) and let \( F \) be a bifunction of \( C \times C \) into \( \mathbb{R} \) satisfying (A1)-(A4). Let \( r > 0 \) and \( x \in H \). Then, there exists \( z \in C \) such that

\[
F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \quad \text{for all} \quad y \in C.
\]

The following lemma was also given in [3].

**Lemma 4.2.** [3] Assume that \( F : C \times C \rightarrow \mathbb{R} \) satisfies (A1)-(A4). For \( r > 0 \) and \( x \in H \), define a mapping \( T_r : H \rightarrow C \) as follows:

\[
T_r(x) = \{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \}
\]

for all \( z \in H \). Then, the following hold:

1. \( T_r \) is single-valued;
2. \( T_r \) is firmly nonexpansive, i.e., for any \( x, y \in H \), \( \|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle \);
3. \( F(T_r) = EP(F) \);
4. \( EP(F) \) is closed and convex.

**Lemma 4.3.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( A : C \rightarrow H \) be a continuous monotone mapping, \( \psi : C \rightarrow \mathbb{R} \) a lower semi-continuous and convex function and \( F : C \times C \rightarrow \mathbb{R} \) be a bifunction satisfying (A1)-(A4). Let \( r > 0 \) and \( x \in H \). Then

\( I \) there exists \( z \in C \) such that

\[
F(z, y) + \langle Az, y - z \rangle + \psi(y) - \psi(z) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C. \quad (4.1)
\]

\( II \) If we define a mapping \( K_r : C \rightarrow C \) as follows:

\[
K_r(x) = \{ z \in C : F(z, y) + \langle Az, y - z \rangle + \psi(y) - \psi(z) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \}
\]

for all \( z \in H \). Then, the following hold:
(1) $K_r$ is single-valued;

(2) $K_r$ is firmly nonexpansive, i.e., for any $x, y \in H$, 
    $\|K_rx - K_ry\|^2 \leq \langle K_rx - K_ry, x - y \rangle$;

(3) $F(K_r) = GMEP(F, A, \psi)$;

(4) $GMEP(F, A, \psi)$ is closed and convex.

**Proof.** Define a bifunction $\Theta : C \times C \rightarrow \mathbb{R}$ as follows:

$$\Theta(x, y) = F(x, y) + \langle Ax, y - x \rangle + \psi(y) - \psi(x), \forall x, y \in C.$$ 

Next we prove that $F$ satisfies the conditions (A1) - (A4).

(i) In fact, since 
    $$\Theta(x, x) = F(x, x) + \langle Ax, x - x \rangle + \psi(x) - \psi(x) = 0, \forall x \in C,$$
the condition (A1) is satisfied.

(ii) Since $F$ satisfies the condition (A2), $\psi$ is lower semi-continuous and convex function and $A : C \rightarrow H$ is a continuous monotone mapping, for any $x, y \in C$, we have 

$$\Theta(x, y) + \Theta(y, x) = F(x, y) + F(y, x) + \langle Ax, y - x \rangle + \langle Ay, x - y \rangle + \psi(y) - \psi(x) + \psi(y) - \psi(x) \leq 0.$$

The condition (A2) is proved.

(iii) Since $A$ is continuous and monotone, $\psi$ is convex and lower semi-continuous and $F$ satisfies the condition (A3), we have 

$$\limsup_{t \downarrow 0} \Theta(x + t(u - x), y) = \limsup_{t \downarrow 0} \left\{ F(x + t(u - x), y) + \langle A(x + t(u - x)), y - (x + t(u - x)) \rangle + \psi(y) - \psi(x + t(u - x)) \right\} \leq F(x, y) + \limsup_{t \downarrow 0} \left\{ \langle A(x + t(u - x)), y - (x + t(u - x)) \rangle \right\} + \psi(y) - \liminf_{t \downarrow 0} \psi(x + t(u - x)) \leq F(x, y) + \limsup_{t \downarrow 0} \left\{ \langle A(x + t(u - x)), y - (x + t(u - x)) \rangle \right\} + \psi(y) - \psi(x) = \Theta(x, y).$$

The condition (A3) is proved.

(iv) By the assumption that the function $y \mapsto F(x, y)$ and $\psi$ both are convex and lower semi-continuous. Again since the function $y \mapsto \langle Ax, y - x \rangle$
is convex and continuous, thus the function \( y \mapsto \Theta(x, y) \) is convex and lower semi-continuous, i.e., \( \Theta \) satisfies the condition (A4).

Hence the conclusions (I) and (II) of Lemma 4.3 can be obtained from Lemma 4.2 immediately.

**Theorem 4.4.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( A : C \rightarrow H \) be a continuous monotone mapping, \( \psi : C \rightarrow \mathbb{R} \) a lower semi-continuous and convex function and \( F : C \times C \rightarrow \mathbb{R} \) be a bifunction satisfies (A1)-(A4). Let \( S \) be a nonspreading mapping of \( C \) into itself such that \( F(S) \cap \text{GMEP}(F, A, \psi) \neq \emptyset \). Suppose \( x_0 = x \in C \) and define the sequence \( \{x_n\} \) and \( \{y_n\} \) by

\[
\begin{align*}
F(y_n, y) + \langle Ay_n, y - y_n \rangle + \psi(y) - \psi(y_n) + \frac{1}{r_n} \langle y - y_n, y_n - x_n \rangle & \geq 0, \quad \forall y \in C, \\
x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Sy_n,
\end{align*}
\]

for all \( n \in \mathbb{N} \), where \( \{r_n\} \in (0, \infty) \) with \( \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty \) and \( \{\alpha_n\} \subseteq [a, b] \) for some \( a, b \in (0, 1) \). Then \( \{x_n\} \) converges weakly to \( \hat{z} \in F(S) \cap \text{GMEP}(F, A, \psi) \).

**Proof.** Setting \( T_n \equiv T_{r_n} \) in Theorem 3.1 then we have \( y_n = T_{r_n} x_n \). Let \( v \in F(S) \cap \text{GMEP}(F, A, \psi) \). For \( n \in \mathbb{N} \), let \( z_n = T_{r_n} z \). We first prove that

\[
\sum_{n=1}^{\infty} \sup \left\{ \|T_{r_{n+1}} z - T_{r_n} z\| : z \in C \right\} < \infty \quad (4.3)
\]

We note that

\[
F(z_n, y) + \langle Az_n, y - z_n \rangle + \psi(y) - \psi(z_n) + \frac{1}{r_n} \langle y - z_n, z_n - z \rangle \geq 0 \quad (4.4)
\]

for all \( y \in C \) and

\[
F(z_{n+1}, y) + \langle Az_{n+1}, y - z_{n+1} \rangle + \psi(z_{n+1}) - \psi(z_n) + \frac{1}{r_{n+1}} \langle y - z_{n+1}, z_{n+1} - z \rangle \geq 0 \quad (4.5)
\]

for all \( y \in C \). Setting \( y = z_{n+1} \) in (4.4) and \( y = z_n \) in (4.5), we have

\[
F(z_n, z_{n+1}) + \langle Az_n, z_{n+1} - z_n \rangle + \psi(z_{n+1}) - \psi(z_n) + \frac{1}{r_n} \langle z_{n+1} - z_n, z_n - z \rangle \geq 0
\]

and

\[
F(z_{n+1}, z_n) + \langle Az_{n+1}, z_n - z_{n+1} \rangle + \psi(z_n) - \psi(z_{n+1}) + \frac{1}{r_{n+1}} \langle z_n - z_{n+1}, z_{n+1} - z \rangle \geq 0.
\]
Adding the two inequalities and by (A2), we have
\[
\langle Az_n - Az_{n+1}, z_{n+1} - z_n \rangle + \left\langle z_{n+1} - z_n, \frac{z_n - z}{r_n} - \frac{z_{n+1} - z}{r_{n+1}} \right\rangle \geq 0.
\]
Thus, we have
\[
\left\langle z_{n+1} - z_n, \frac{z_n - z}{r_n} - \frac{z_{n+1} - z}{r_{n+1}} \right\rangle \geq \langle Az_{n+1} - Az_n, z_{n+1} - z_n \rangle
\]
and hence
\[
\langle z_{n+1} - z_n, z_{n+1} - z_n \rangle + \left(1 - \frac{r_n}{r_{n+1}}\right) (z_{n+1} - z) \geq \langle Az_{n+1} - Az_n, z_{n+1} - z_n \rangle
\]
Since \(A\) is continuous monotone mapping, we have
\[
-\|z_{n+1} - z_n\|^2 + \left\langle z_{n+1} - z_n, \left(1 - \frac{r_n}{r_{n+1}}\right) (z_{n+1} - z) \right\rangle \geq 0.
\]
So, we get
\[
\|z_{n+1} - z_n\|^2 \leq \left\langle z_{n+1} - z_n, \left(1 - \frac{r_n}{r_{n+1}}\right) (z_{n+1} - z) \right\rangle \leq \|z_{n+1} - z_n\| \left|1 - \frac{r_n}{r_{n+1}}\right| \|z_{n+1} - z\|.
\]
Without loss of generality, let us assume that there exists a real number \(b\) such that \(r_n > b > 0\) for all \(n \in \mathbb{N}\). Then
\[
\|T_{r_{n+1}} z - T_r z\| = \|z_{n+1} - z_n\| \leq \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|T_{r_{n+1}} z - z\| \leq \frac{1}{b} |r_{n+1} - r_n| \|T_{r_{n+1}} z - z\|. \tag{4.6}
\]
Let \(u \in GMEP(F, A, \psi)\) and \(M = \sup \{\|z - u\|: z \in C\}\). Then
\[
\|T_{r_{n+1}} z - z\| \leq \|T_{r_{n+1}} z - u\| + \|u - z\| = \|T_{r_{n+1}} z - T_{r_{n+1}} u\| + \|u - z\| \leq 2\|z - u\|.
\]
This together with (4.6), we have
\[
\sup \|T_{r_{n+1}} z - T_r z\| : z \in C \leq \frac{2M}{b} |r_{n+1} - r_n|.
\]
Since $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, we obtain $\sum_{n=1}^{\infty} \sup \{ \| T_{r_{n+1}} z - T_{r_n} z \| : z \in C \} < \infty$. By Lemma 2.4, we define a mapping $T$ by $Tx = \lim_{n \to \infty} T_{r_n}x$ for all $x \in C$.

Next, we prove that $F(T) = \cap_{n=1}^{\infty} F(T_{r_n})$. It easy to see that $\cap_{n=1}^{\infty} F(T_{r_n}) \subset F(T)$. Let $w \in F(T)$. For $n \in \mathbb{N}$, let $w_n = T_{r_n}w$. Then

$$F(w_n, y) + \langle Aw_n, y - w_n \rangle + \psi(y) - \psi(w_n) + \frac{1}{r_n} \langle y - w_n, w_n - w \rangle \geq 0$$

for all $y \in C$. By (A2), we obtain $\frac{1}{r_n} \langle y - w_n, w_n - w \rangle \geq F(y, w_n) + \langle Aw_n, w_n - y \rangle - \psi(y) + \psi(w_n)$ for all $y \in C$. Since $w_n \to w$, $A$ is continuous monotone mapping, $\psi$ is lower semi-continuous mapping and from (A4), we have $0 \geq F(y, w) + \langle Aw, w - y \rangle - \psi(y) + \psi(w)$ for all $y \in C$. Put $u_t = ty + (1-t)w$ for all $t \in (0,1]$ and $y \in C$. Thus, we note that

$$0 = F(u_t, u_t) + \langle Aw, u_t - u_t \rangle + \psi(u_t) - \psi(u_t)$$

$$= F(ty + (1-t)w, ty + (1-t)w) + \langle Aw, (ty + (1-t)w) - u_t \rangle$$

$$\leq tF(ty + (1-t)w, y) + (1-t)F(ty + (1-t)w, w) + t\langle Aw, y - u_t \rangle$$

$$+ (1-t)\langle Aw, w - u_t \rangle$$

$$= t[F(ty + (1-t)w, y) + \langle Aw, y - u_t \rangle]$$

$$+ (1-t)[F(ty + (1-t)w, w) + \langle Aw, w - u_t \rangle]$$

$$\leq t[F(ty + (1-t)w, y) + \langle Aw, y - u_t \rangle].$$

So, $F(ty + (1-t)w, y) + \langle Aw, y - u_t \rangle \geq 0$ for all $y \in C$. Letting $t \to 0^+$ and using (A3), $A$ is continuous monotone mapping. So, we obtain $F(w, y) + \langle Aw, y - w \rangle \geq 0$ for all $y \in C$. Thus $w \in GMEP(F, A, \psi)$. It follows that $w \in \cap_{n=1}^{\infty} F(T_{r_n})$. Hence $\{T_{r_n}\}$ satisfy condition in Theorem 3.1. So, we obtain the desired result by using Theorem 3.1. \hfill \Box

**Acknowledgement.** The authors would like to thank the referees for the insightful comments and suggestions. Moreover, the authors gratefully acknowledge the Thailand Research Fund Master Research Grants (TRF-MAG, MRG-WI515S029) for funding this paper.

**References**


Received: April, 2011