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An Analytic Algorithm for Generalized Abel Integral Equation

Sunil Kumar, Om P. Singh* and Sandeep Dixit

Department of Applied Mathematics, Institute of Technology
Banaras Hindu University, Varanasi -221005, India

Abstract. In this paper, a homotopy perturbation method (HPM) and modified homotopy perturbation (MHPM) are proposed to solve singular integral equation with generalized Abel's kernel.

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1. Introduction

The real world problems in scientific fields such as solid state physics, plasma physics, fluid mechanics, chemical kinetics and mathematical biology are nonlinear in general when formulated as partial differential equations or integral equations. In the last two decades, many powerful and simple methods have been proposed and applied successfully to approximate various types of singular integral equations with a wide range of applications [1,2,6-8].

Corresponding author. Email: *singhom@gmail.com (O. P. Singh), skiitbhu28@gmail.com (S. Kumar)

The generalized Abel's integral equation of the second kind is given by

$$y(x) = f(x) + \lambda \int_0^x \frac{y(t)}{(x-t)^\alpha} dt, \quad 0 \le x \le 1, \tag{1}$$

where $\lambda \in C$ is a parameter and $0 < \alpha < 1$. A well known result for the above equation is the following.

Theorem 1: For each complex $\lambda \neq \infty$,

$$y(x) = f(x) + \int_0^x R(x-t) f(t) dt, \tag{2}$$

where $R(x) = \sum_{n=1}^{\infty} \frac{\left[\lambda \Gamma(1-\alpha) x^{(1-\alpha)}\right]^n}{x \Gamma\left[n(1-\alpha)\right]},$ (3)

is the unique solution of equation (1), [4].

The closed form solution (2) is not very useful in many cases where it is difficult to evaluate the integral appearing in (2). So, it is desirable to have numerical solution for the generalized Abel's integral equation (1).

Recently, Pandey et al [7] obtained numerical solution of (1) for $\lambda = 1$ and $\alpha = \frac{1}{2}$, using HPM, ADM, MHPM and MADM. This motivated us for the present work where we have taken α to be any number between 0 and 1.

The aim of the present paper is to propose HPM to solve the generalized Abel's integral equation (1) and show that ADM and MADM is HPM and MHPM with the constructed convex homotopy to solve (1) respectively.

2. Homotopy Perturbation Method and its modification

In this method, using the homotopy technique of topology, a homotopy is constructed with an embedding parameter $p \in [0,1]$ which is considered as a "small parameter". This method became very popular amongst the scientists and engineers, even though it involves continuous deformation of a simple problem into a more difficult problem under consideration. Most of the perturbation methods depend on the existence of a small perturbation parameter but many nonlinear problems have no small perturbation parameter at all. Many new methods have been proposed in the late nineties to solve such nonlinear equation devoid of such small parameters. Late 1990s saw a surge in applications of homotopy theory in the scientific and engineering computations $[1,2,5-7]$. When the homotopy theory is coupled with perturbation theory it provides a powerful mathematical tool. A review of recently developed methods of nonlinear analysis can be found in [6]. To illustrate the basic concept of HPM, consider the following nonlinear functional equation

$$A(u) = f(r), \quad r \in \Omega, \text{ with the boundary conditions } B\left(u, \frac{\partial u}{\partial n}\right) = 0, \ \ r \in \partial\Omega, \tag{4}$$

where A is a general functional operator, B is a boundary operator, $f(r)$ is a known analytic function, and $\partial\Omega$ is the boundary of the domain Ω. The operator A is decomposed as $A = L + N$, where L is the linear and N is the nonlinear operator. Hence Equation (4) can be written as

$$L(u) + N(u) - f(r) = 0, \quad r \in \Omega.$$

We construct a homotopy $v(r, p) : \Omega \times [0,1] \to R$ satisfying

$$H(v, p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \qquad p \in [0,1], \qquad r \in \Omega. \tag{5}$$

Hence

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{6}$$

where u_0 is an initial approximation for the solution of (4). As

$$H(v,0) = L(v) - L(u_0) \ \ \textit{and} \ \ H(v,1) = A(v) - f(r), \tag{7}$$

it shows that $H(v, p)$ continuously traces an implicitly defined curve from a starting point $H(u_0,0)$ to a solution $H(v,1)$. The embedding parameter p increases monotonously from zero to one as the trivial linear part $L(u) = 0$ deforms continuously to the original problem $A(u) = f(r)$. The embedding parameter $p \in [0,1]$ can be considered as an expanding parameter [5] to obtain

$$v = v_0 + pv_1 + p^2 v_2 + \dots . \tag{8}$$

The solution is obtained by taking the limit as p tends to 1 in equation. (8). Hence

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \dots . \tag{9}$$

The series (9) converges for most cases and the rate of convergence depends on $A(u) - f(r)$ [6].

3. Method of Solution

We consider the following convex homotopy

$$(1-p)\left[L(x)-y_0(x)\right]+p\left[L(x)-f(x)-\int_0^x \frac{L(t)}{(x-t)^{\alpha}}dt\right]=0, \tag{10}$$

We seek the solution of (1) in the following form,

$$L(x)=\lim_{n\to\infty}\sum_{i=0}^{n} p^i L_i(x) \tag{11}$$

where, $L_i(x), i=0,1,2,3,...$ is the function to be determined. We use the following iterative scheme to evaluate $L_i(x)$. The initial approximation to the solution $L_0(x)=y_0(x)$ is taken to be $f(x)$, therefore, $L_0(x)=y_0(x)=f(x)$.

Substituting (11) in (10) and equating the corresponding powers of p, we get

$$p^0: \quad L_0(x)=f(x),$$

$$p^1: \quad L_1(x)=\int_0^x \frac{L_0(t)}{(x-t)^{\alpha}}dt,$$

$$p^2: \quad L_2(x)=\int_0^x \frac{L_1(t)}{(x-t)^{\alpha}}dt,$$

$$p^3: \quad L_3(x)=\int_0^x \frac{L_2(t)}{(x-t)^{\alpha}}dt,$$

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$$p^n: \; L_n(x)=\int_0^x \frac{L_{n-1}(t)}{(x-t)^{\alpha}}dt. \tag{12}$$

Hence, the solution of Equation (1) is given by

$$y(x)=\lim_{p\to 1} L(x)=\sum_{i=0}^{\infty} L_i(x) \tag{13}$$

In the modified Homotopy perturbation method (MHPM), we break $f(x)$ into an infinite sum as follows,

$$f(x) = \sum_{i=0}^{\infty} k_i(x), \text{ and define, } \psi(x, p) = \sum_{i=0}^{\infty} p^i k_i(x) \rightarrow f(x) \text{ as } p \rightarrow 1. \qquad (14)$$

The initial approximation to the solution is taken to be $k_0(x)$. Substituting (11) and (14) into (10) and equating coefficients of p with the same power one gets the exact solution.

It is to be noted that the rate of the convergence of the series (14) depends upon the initial choice $y_0(x)$ also as illustrated by the given numerical examples.

4. Adomian Decomposition Method and its modification

The Adomian decomposition method has been applied to a wide class of functional equations [3,9] by scientists and engineers since the beginning of the 1980s. Adomian gives the solution as an infinite series usually converging to a solution. Consider the following singular integral equation of the second kind of the form

$$y(x) = f(x) + \int_0^x k(x,t)y(t)dt, \quad f(x) \in L^2(a,b). \qquad (15)$$

The ADM assumes an infinite series solution for the unknown function $y(x)$, given by

$$y(x) = \sum_{n=0}^{\infty} y_n(x). \qquad (16)$$

Substituting (16) into (15), we get

$$\sum_n y_n(x) = f(x) + \int_0^x k(x,t) \sum_n y_n(t)dt. \qquad (17)$$

The ADM use the following recursive relation to evaluate the various iterates $y_1, y_2, y_3, \ldots$ in (16)

$$y_0(x) = f(x), \qquad y_{n+1}(x) = \int_0^x k(x,t)y_n(t)dt, \quad n \geq 0. \qquad (18)$$

Recently, Wazwaz [9] proposed a modification in ADM by constructing the zeroth component $y_0(x)$ of the decomposition in a slightly different way. He splitted the function $f(x)$ as the sum of two functions $f_1(x)$ and $f_2(x)$ in $L^2(R)$. and suggested the following recursive scheme:

$$y_0(x) = f(x),$$

$$y_1(x) = f_1(x) + \int_0^x k(x,t)y_0(t)dt,$$

$$y_{n+1}(x) = \int_0^x k(x,t)y_n(t)dt, \quad n > 1. \tag{19}$$

This type of modification provides more flexibility to the ADM in solving complicated integral equations and avoids the unnecessary complexity in calculating the Adomian polynomials. In this case, the decomposition series (16) has a rapid rate of convergence in real physical problem. The rapid convergence ensures that only a few iterations are required to get the accurate solution of the problem. In this paper, we assume the kernel $k(x,t)$ to be generalized Abel's kernel i.e.

$$k(x,t) = \frac{1}{(x-t)^{\alpha}}, \qquad 0 \le x \le 1, \quad 0 < \alpha < 1.$$

Theorem2. The Adomian decomposition method is a special case of the He's homotopy perturbation method. Likewise modified Adomian decomposition method follows from the modified homotopy perturbation method.

Proof. Follows trivially from equations (11), (12), (16) and (18).

5. Illustrative Examples

The simplicity and accuracy of the proposed method is illustrated by the following numerical example and by computing the absolute errors $E(x) = | y(x) - y_a(x) |$, where $y(x)$ is the exact solution and $y_a(x)$ is an approximate solution of the problem obtained by truncating equation (11).

Example 1 Consider the Generalized Abel's integral equation of second kind

$$y(x) = x^2 + \frac{27}{40} x^{8/3} - \int_0^x \frac{y(t)}{(x-t)^{1/3}} dt, \tag{20}$$

with the exact Solution $y(x) = x^2$.

Case 1(a) *Homotopy perturbation method*

Taking $L_0(x) = x^2 + \frac{27}{40} x^{8/3}$, the various iterates are obtained from equation (12). The first few iterates are as follows:

$$L_1(x) = -\frac{27}{40} x^{8/3} - \frac{2x^{10/3}\left(\Gamma\left(\frac{2}{3}\right)\right)^2}{\Gamma\left(\frac{13}{3}\right)},$$

$$L_2(x) = \frac{x^4\left(\Gamma\left(\frac{2}{3}\right)\right)^3}{12} + \frac{27x^{10/3}\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{11}{3}\right)}{40\,\Gamma\left(\frac{13}{3}\right)},$$

$$L_3(x) = -\frac{x^4\left(\Gamma\left(\frac{2}{3}\right)\right)^3}{12} - \frac{243x^{14/3}\left(\Gamma\left(\frac{2}{3}\right)\right)^3}{6160}, \ldots$$

The Fig.1 shows the absolute error between the exact solution $y(x)$ and the approximate solution $y_a(x)$ obtained from (11) by truncating it at level n=13.

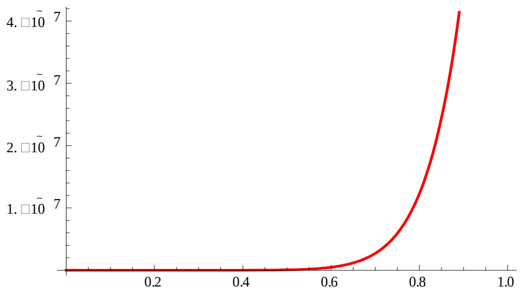

Figure 1. The absolute error for Example1, case 1(a) (n=13).

Case 1(b) Now choosing a different initial guess

$L_0(x) = x,$ the following iterates of the solution are obtained

$$L_1(x) = -x + x^2 - \frac{9}{10} x^{5/3} + \frac{27}{40} x^{8/3},$$

$$L_2(x) = \frac{9}{10}x^{5/3} - \frac{27}{40}x^{8/3} + \frac{x^{7/3}\left(\Gamma\left(\frac{2}{3}\right)\right)^2}{\Gamma\left(\frac{10}{3}\right)} - \frac{3x^{10/3}\left(\Gamma\left(\frac{2}{3}\right)\right)^2}{5\Gamma\left(\frac{10}{3}\right)},\ldots$$

The Fig. (2) is drawn at the same level of truncation, n (=13) as was the case of Figure (1).

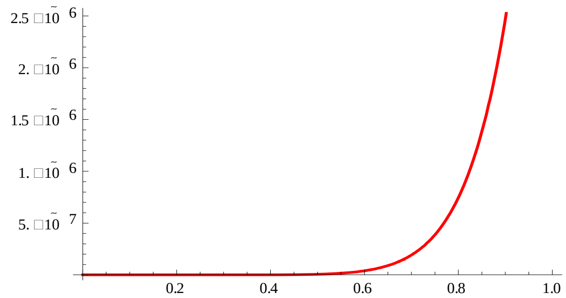

Figure 2. The absolute error for Example 1, case 1(b) (n=13).

From Figs.1 and 2, one observes the dependence of the convergence rate of the series (13) on the initial choice $y_o(x)$.

Case 1(c) *Modified Homotopy perturbation method*

Writing $f(x) = \sum_{i=0}^{\infty} k_i(x)$. where $k_0(x) = x^2, k_1(x) = \frac{27}{40}x^{8/3}$ and $k_i(x) = 0$ for $i \geq 2$, we get $L_0(x) = x^2$.

Hence, the various iterates are as follows:

$$p^0: \quad L_0(x) = x^2,$$

$$p^1: \quad L_1(x) = \frac{27}{40}x^{8/3} - \int_0^x \frac{L_0(t)}{(x-t)^{1/3}}dt = 0,$$

$$p^2: \quad L_2(x) = \int_0^x \frac{L_1(t)}{(x-t)^{1/3}}dt = 0.$$

Therefore, one can see that $L_n(x) = 0$, for all $n \geq 1$, and hence,

$$L(x) = L_0(x) + L_1(x) + L_2(x) + ... = x^2,$$

is the exact solution.

5. Conclusions

We have proposed a simple algorithm based on homotopy perturbation method to solve generalized Abel's integral equation. It is proved that ADM is special case of HPM. From the above numerical example, it is obvious that when we apply MHPM, the iterates become zero from second iterates itself. The rate of convergence for the series representing the solution obtained by HPM depends upon the initial choice $L_0(x)$.

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