A New Criteria for a Matrix is not Generalized Strictly Diagonally Dominant Matrix

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Abstract

In this paper, we obtained some new results about a matrix is not generalized strictly diagonally dominant matrix by using gauss elimination and some exists theories, which improve the corresponding results of [4]. Advantages are illustrated by numerical examples.

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1 Introduction

Generalized strictly diagonally dominant matrix play an important role in numerical algebra, control theory, electric system, economic mathematics and elastic dynamics and so on. But it isn’t easy to determine whether a matrix is or not a generalized strictly diagonally dominant matrix in reality. In this paper, we obtained some new results about a matrix is not generalized strictly diagonally dominant matrix.

2 Preliminary Notes

Let $C^{n \times n}(R^{n \times n})$ be the set of complex (real) $n \times n$ matrices, $A = (a_{ij}) \in C^{n \times n}$, $N = \{1, 2, \cdots, n\}$, for any $i, j \in N$, we denote:

$$\alpha_i = \sum_{j \in N, j \neq i} |a_{ij}|, \beta_i = \sum_{j \not\in N, j \neq i} |a_{ij}|, \tilde{\alpha}_i = \sum_{j \in N, j \neq i} |a_{ji}|, \tilde{\beta}_i = \sum_{j \not\in N, j \neq i} |a_{ji}|,$$

$$\Lambda_i = \sum_{j \neq i} |a_{ij}| = \alpha_i + \beta_i, S_i = \sum_{j \neq i} |a_{ji}| = \tilde{\alpha}_i + \tilde{\beta}_i,$$

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where \( N = N_1 \oplus N_2, N_1 = \{ i | |a_{ii}| > \Lambda_i, i \in N \}, N_2 = \{ i | 0 < |a_{ii}| \leq \Lambda_i(A), i \in N \} \). In this paper, we denote \( A^{(k)} = (a^{(k)}_{ij}) \) is a matrix that \( A \) after \( k \)th steps gauss elimination.

**Definition 2.1** Let \( A = (a_{ij}) \in C^{n \times n} \).

(i) If \( |a_{ii}| \geq \Lambda_i, \forall i \in N \), then \( A \) is a diagonally dominant matrix;

(ii) If \( |a_{ii}| > \Lambda_i, \forall i \in N \), then \( A \) is a strictly diagonally dominant matrix, we denote it by \( A \in D_0 \);

(iii) If there exists a positive diagonal matrix \( D = \text{diag}\{d_1, d_2, \ldots, d_n\} \), such that \( AD = B \in D_0 \), then \( A \) is a generalized strictly diagonally dominant matrix, we denote it by \( A \in \tilde{D} \).

It is obvious that if \( N_1 = \Phi \) be an empty set, then \( A \notin \tilde{D} \), if \( N_2 = \Phi \), then \( A \in D^* \).

**Definition 2.2** Let \( A = (a_{ij}) \in C^{n \times n} \).

(i) If there exists an \( \alpha \in (0, 1] \) and \( |a_{ii}| \geq \alpha \Lambda_i + (1 - \alpha)S_i \) (1.1) for any \( i \in N \), then \( A \) is an \( \alpha \) diagonally dominant matrix. We denote it by \( A \in D_0(\alpha) \);

(ii) If all strictly inequality holds in (1.1), then \( A \) is a strictly \( \alpha \) diagonally dominant matrix. We denote it by \( A \in D(\alpha) \);

(iii) If there exists a positive diagonal matrix \( D \), such that \( AD = B \in D_0 \), then \( A \) is a generalized strictly \( \alpha \) diagonally dominant matrix. We denote it by \( A \in D^*(\alpha) \).

**Definition 2.3** Let \( A = (a_{ij}) \in C^{n \times n} \).

(i) If there exists an \( \alpha \in (0, 1] \) and \( |a_{ii}a_{jj}| \geq (\Lambda_i\Lambda_j)\alpha(S_iS_j)^{1-\alpha} \) (1.2) for any \( i \neq j, (i, j \in N) \), then \( A \) is an \( \alpha \) bi-diagonally dominant matrix. We denote it by \( A \in D_0^\alpha \);

(ii) If all strictly inequality holds in (1.2), then \( A \) is a strictly \( \alpha \) bi-diagonally dominant matrix. We denote it by \( A \in D^\alpha \);

(iii) If there exists a positive diagonal matrix \( D \), such that \( AD = B \in D^\alpha \), then \( A \) is a generalized strictly \( \alpha \) bi-diagonally dominant matrix. We denote it by \( A \in \tilde{D}^\alpha \).

**Definition 2.4** Let \( A = (a_{ij}) \in C^{n \times n} \), if \( \mu(A) = (m_{ij}) \in R^{n \times n} \),

\[
m_{ij} = \begin{cases} 
|a_{ij}|, & i = j, \\
-|a_{ij}|, & i \neq j.
\end{cases}
\]

then we say \( \mu(A) \) is a comparison matrix of \( A \).

A matrix \( A = (a_{ij}) \in C^{n \times n} \) is called nonsingular H-matrix if its comparison matrix \( \mu(A) \) is a nonsingular M-matrix.
3 Main Results

In this section, we will give some criteria for a matrix is not generalized strictly diagonally dominant matrix.

Lemma 3.1 ([1]) If \( A \in D(\alpha) \), then \( \mu(A) \) is a nonsingular M-matrix and \( A \in D^* \).

Lemma 3.2 ([2]) If \( A \in D^\alpha \), then \( A \in D^\alpha \).

Lemma 3.3 ([3]) Let \( A = (a_{ij}) \in C^{n \times n} \), if \( A \) is a nonsingular H-matrix, then there exists at least one strict diagonally dominant row, namely \( N_1 \neq \Phi \).

Lemma 3.4 ([4]) Let \( A = (a_{ij}) \in C^{n \times n} \), if there exist \( N_1, N_2 \) such that \( N_1 \cup N_2 = N \), and

\[
(|a_{ii} - \alpha_i)(|a_{jj}| - \beta_j) \leq \beta_i \alpha_j
\]

for any \( i \in N_1, j \in N_2 \), then \( A \) is not a generalized strictly diagonally dominant matrix and \( \mu(A) \) is not a nonsingular M-matrix.

Theorem 3.5 Let \( A = (a_{ij}) \in C^{n \times n} \), if \( A \in D_0 \), then \( A^{(n-1)} \in D_0 \).

Proof. As \( A \in D_0 \), we have \( a_{11} \neq 0 \) and

\[
|a_{ii}| > \sum_{j \neq i} |a_{ij}| = |a_{i1}| + \sum_{j \neq 1, i} |a_{ij}|
\]

\[
> |a_{i1}| \frac{A_1}{|a_{11}|} + \sum_{j \neq 1, i} |a_{ij}|
\]

\[
= \frac{|a_{i1}|}{|a_{11}|} \sum_{j \neq 1, i} |a_{1j}| + \frac{|a_{1i}|}{|a_{11}|} |a_{11}| + \sum_{j \neq 1, i} |a_{ij}|
\]

it follows that

\[
|a_{ii}| - \frac{|a_{i1}|}{|a_{11}|} |a_{11}| > \sum_{j \neq 1, i} |a_{ij}| + \sum_{j \neq 1, i} \frac{|a_{i1}|}{|a_{11}|} |a_{1j}|
\]

for matrix \( A^{(1)} \).

(i) When \( i = 1 \). It is obvious that

\[
|a_{11}^{(1)}| = |a_{11}| > \sum_{j \neq 1} |a_{1j}| = \sum_{j \neq 1} |a_{1j}^{(1)}|
\]
(ii) When $1 < i \leq n$.

\[
|a_{ii}^{(1)}| = |a_{ii} - \frac{a_{i1}a_{i1}}{a_{11}}| > |a_{ii} - \frac{a_{i1}}{a_{11}}a_{11}|
\]

\[
> \sum_{j \neq i} |a_{ij}| + \sum_{j \neq i} \frac{|a_{i1}|a_{1j}}{|a_{11}|}
\]

\[
> \sum_{j \neq i} |a_{ij} - \frac{a_{i1}a_{1j}}{a_{11}}|
\]

\[
= \sum_{j \neq i} |a_{ij}^{(1)}| = \Lambda_i(A^{(1)})
\]

thus $A^{(1)} \in D_0$, in the same way, after $n - 1$ steps gauss elimination, we have $A^{(n-1)} \in D_0$.

**Theorem 3.6** Let $A = (a_{ij}) \in C^{n \times n}$, if $A \in D^*$, then $A^{(n-1)} \in D^*$.

**Proof.** As $A \in D^*$, there exists a positive diagonal matrix $D$ satisfied $AD \triangleq B \in D_0$. We construct the inverse Frobenius matrix as follow:

\[
L_1^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
-c_{21} & 1 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-c_{n1} & \cdots & \cdots & 1
\end{pmatrix}
\]

where $c_{ii} = \frac{a_{ii}}{a_{11}}$, $(i = 2, 3, \cdots, n)$, since $L_1^{-1}B = L_1^{-1}(AD) = (L_1^{-1}A)D = A^{(1)}D$, by Theorem 3.1 we know $A^{(1)}D \in D_0$, therefore $A^{(1)} \in D^*$, in the same way, we have $A^{(n-1)} \in D^*$.

**Theorem 3.7** Let $A = (a_{ij}) \in C^{n \times n}$, if $A \in D^*(\alpha)$, then $A^{(n-1)} \in D^*$.

**Proof.** By definition 2.2 and Lemma 3.1 we know that there exists a positive diagonal matrix $D_1$ satisfied $AD_1 \in D^*$, furthermore, there exists a positive diagonal matrix $D_2$ such that $AD_1D_2 \triangleq AD \triangleq B \in D_0$, by theorem 3.2, we have $A^{(n-1)} \in D^*$.

**Theorem 3.8** Let $A = (a_{ij}) \in C^{n \times n}$, if $A \in D^\alpha$, then $A^{(n-1)} \in D^*$.

**Proof.** By Lemma 3.2 and Theorem 3.2 we can obtain the result.

**Theorem 3.9** Let $A = (a_{ij}) \in C^{n \times n}$, if $A \in \widetilde{D}^\alpha$, then $A^{(n-1)} \in D^*$.

**Proof.** By definition 1.3 we know that there exists a positive diagonal matrix $D_1$ satisfied $AD_1 \in D^\alpha$, then by Lemma 2.2, $AD_1 \in D^*$, furthermore, there exists a positive diagonal matrix $D_2$ such that $AD_1D_2 \triangleq AD \triangleq B \in D_0$, by Theorem 3.2, we have $A^{(n-1)} \in D^*$.

**Theorem 3.10** Let $A = (a_{ij}) \in C^{n \times n}$, if $A^{(n-1)} \not\in D^*$, then $A \not\in D^*$.

**Proof.** Suppose $A \in D^*$, by Theorem 3.2 we have $A^{(n-1)} \in D^*$, the results conflicts with the conditions of this theorem, thus $A \not\in D^*$. 
4 Numerical example

In this section, we give some examples to illustrate our results is effective.

**Example 4.1** Let

\[
A = \begin{pmatrix}
1 & 0.5 & 0.4 \\
0.3 & 1 & 0.5 \\
0.1 & 0.6 & 1
\end{pmatrix}
\]

obviously \( A \in D_0 \), since

\[
A^{(2)} = \begin{pmatrix}
1 & 0.5 & 0.4 \\
0 & 0.85 & 0.38 \\
0 & 0 & 0.607 \div 0.850
\end{pmatrix}
\]

so \( A^{(2)} \in D_0 \).

**Example 4.2** Let

\[
A = \begin{pmatrix}
1 & 0.4 & 0.2 & 0.1 \\
0.3 & 1 & 0.1 & 0.2 \\
0 & 0.5 & 1 & 0.4 \\
0.8 & 0.4 & 0.5 & 1
\end{pmatrix}
\]

when \( \alpha = \frac{1}{2}, \forall i, j \in \{1, 2, 3, 4\}, (i \neq j) \), we have \(|a_{ii}a_{jj}| \geq (A_iA_j)^{\frac{1}{2}}(S_iS_j)^{\frac{1}{2}}\), by definition 2.3, \( A \in D^\alpha \), since

\[
A^{(3)} = \begin{pmatrix}
1 & 0.4 & 0.2 & 0.1 \\
0 & 0.88 & 0.04 & 0.17 \\
0 & 0 & 0.9773 & 0.3034 \\
0 & 0 & 0 & 0.8001
\end{pmatrix}
\]

so \( A^{(3)} \in D_0 \).

**Example 4.3** Let

\[
A = \begin{pmatrix}
1 & 0.1 & 1 \\
0 & 1 & 1.4 \\
1 & 1.1 & 2.4
\end{pmatrix}
\]

since

\[ (|a_{33}| - \alpha_3)(|a_{11}| - \beta_1) = 2.16 > 2.1 = \beta_3\alpha_1 \]

so \( A \) does not satisfy the conditions of the Theorem 3 in [4], that is the Lemma 3.4 in this paper, but

\[
A^{(2)} = \begin{pmatrix}
1 & 0.1 & 1 \\
0 & 1 & 1.4 \\
0 & 0 & 0
\end{pmatrix}
\]

so \( A \) does not satisfy Lemma 3.3, \( A^{(2)} \not\in D^\ast \), by Theorem 3.6, we have \( A \not\in D^\ast \).

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References


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