Legendre Wavelets Method for Fractional Integro-Differential Equations

E. A. Rawashdeh

Department of Mathematics and Sciences
Dhofar University, Salalah Oman
edris@du.edu.om

Abstract

Legendre wavelets methods are commonly used for the numerical solution of integral equations. In this paper, we apply the Legendre wavelets method to approximate the solution of fractional integro-differential equations. Numerical examples are also presented to demonstrate the validity of the method.

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1 Introduction.

We extend the iterative algorithms due to S. Yousefi and M. Razzaghi [11] for computing the sequence \( \{y_{k,M}(t)\} \) of the truncated wavelets series

\[
y_{k,M}(t) = \sum_{n=1}^{2^{k-1}M-1} \sum_{m=0}^{M-1} c_{nm} \psi_{n,m}(t), \quad t \in [0,1]
\]

interpolating the exact solution \( y(t) \) of the initial-value problem for fractional integro-differential equation

\[
D_0^q y(t) = f(t) + p(t)y(t) + \int_0^t k(t,s)y(s)ds, \quad t \in I = [0,1], \quad y(0) = \alpha \quad (1)
\]
where $0 < q < 1$ and $\alpha \in \mathbb{R}$. Here, the given functions $f, p, k$ are assumed to be sufficiently smooth on their respective domains $I$ and $S$ ($S = \{(t, s) : 0 \leq s \leq t \leq 1\}$). In equation (1), $\psi_{n,m}(t)$ is the Legendre wavelets polynomial and $D_q^*$ denotes Caputo fractional derivative of order $q$. Amongst a variety of definition for fractional order derivatives, Caputo fractional derivative has been used because it allows physically interpretable initial conditions; see for example [5]. Recently some attentions have been paid to the numerical solution of equation (1). Rawashdeh [8] applied the collocation method to find a spline approximation, Shaer and algarellh [6] used the decomposition method to find an analytic solution, and A. Arikoglu and I. Ozkol [1] extended the fractional transform method which is a semi analytical numerical technique to approximate the solution of equation (1).

Equation (1) occurs in a wide variety of applied sciences. It is encountered in various areas of physics and engineering [7, 9]. Damping laws, diffusion processes [2] and fractals [9] are better formulated with the use of fractional derivatives.

The outline of this paper is as follows. In Section 2, we present some definitions and preliminaries. Section 3 contains the application of the Legendre wavelets method to equation (1). Some numerical examples are provided in Section 4.

## 2 Definitions and Preliminaries

**Definition 2.1.** A real function $f(x), x > 0$, is said to be in the space $C_\alpha, \alpha \in \mathbb{R}$ if there exist a real number $p > \alpha$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty)$.

**Definition 2.2.** A real function $f(x), x > 0$, is said to be in the space $C_k^\alpha, k \in \mathbb{N} \cup \{0\}$, if $f^{(k)} \in C_\alpha$.

**Definition 2.3.** The Riemann-Liouville fractional integral of order $q \geq 0$ of a function $f \in C_\alpha, \alpha \geq -1$, is defined by

$$J^q f(x) = \frac{1}{\Gamma(q)} \int_0^x (x-s)^{q-1} f(s)ds.$$  

**Definition 2.4.** Let $f \in C_{k-1}^k, k \in \mathbb{N}$. Then the Caputo fractional derivative of $f$ is defined by

$$D_q^* f(x) = \begin{cases} J^{k-q} f^{(k)}(x), & \text{if } k-1 < q \leq k, \\ f^{(k)}(t), & \text{if } q = k. \end{cases}$$
Some important properties of the Caputo operator can be found in [7].

To obtain a numerical scheme for the approximation of Caputo derivative, we can use a representation that has been introduced by Elliot [4];

$$D_s^{(q)} f(x) = \frac{1}{\Gamma(-q)} \int_0^x \frac{f(s) - f(0)}{(x - s)^{1+q}} ds, \quad (2)$$

where the integral in equation (2) is a Hadamard finite-part integral.

**Definition 2.5.** The following functions

$$\psi_{k,n}(t) = |a_0|^{k/2} \psi(a_0^k t - nb_0),$$

form a family of discrete wavelets, where $a_0 > 1, b_0 > 0$ and $n, k$ are positive integers and $\psi$ is a given function called mother wavelet. Moreover, the functions

$$\psi_{n,m}(t) = \begin{cases} \sqrt{m + \frac{1}{2}} 2^{k/2} p_m(2^k t - \hat{n}), & \frac{\hat{n}-1}{2^k} \leq t < \frac{\hat{n}+1}{2^k}, \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

are called Legendre wavelets polynomials where $\hat{n} = 2n - 1, n = 1, \ldots, 2^{k-1}, k \in \mathbb{N}$, $t \in [0, 1]$ and $m$ is the order of the Legendre polynomials $p_m$.

## 3 Legendre wavelets method

In the present paper, we consider the following fractional integro-differential equation

$$D_s^\alpha y(t) = f(t) + p(t) y(t) + \int_0^t k(t, s) y(s) ds, \quad t \in I = [0, 1], \quad y(0) = \alpha. \quad (4)$$

The exact solution of equation (4) can be expanded as a Legendre wavelets series as follows:

$$y(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{n,m}(t).$$

where $\psi_{n,m}(t)$ is given by equation (3). We approximate the solution $y(t)$ by the truncated series

$$y_{k,M}(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \psi_{n,m}(t). \quad (5)$$
Then a total number of $2^{k-1}M$ conditions should exist for determination of $2^{k-1}M$ coefficients

$$c_{10}, c_{11}, \ldots, c_{1M-1}, c_{20}, \ldots, c_{2M-1}, \ldots, c_{2^{k-1}0}, \ldots, c_{2^{k-1}M-1}.$$ 

Since one condition is furnished by the initial condition, namely

$$y_{k,M}(0) = \sum_{n=1}^{2^{k-1}M-1} \sum_{m=0}^{M-1} c_{nm} \psi_{n,m}(0) = \alpha. \quad (6)$$

We see that there should be $2^{k-1}M - 1$ extra conditions to recover the unknown coefficients $c_{nm}$. These conditions can be obtained by substituting equation (5) in equation (4);

$$\sum_{n=1}^{2^{k-1}M-1} \sum_{m=0}^{M-1} c_{nm} D_s^q \psi_{n,m}(t) = f(t) + \sum_{n=1}^{2^{k-1}M-1} \sum_{m=0}^{M-1} c_{nm} p(t) \psi_{n,m}(t) \quad (7)$$

$$+ \sum_{n=1}^{2^{k-1}M-1} \sum_{m=0}^{M-1} c_{nm} \int_0^t k(t, s) \psi_{n,m}(s) ds. \quad (8)$$

We now assume equation (7) is exact at $2^{k-1}M - 1$ points $x_i$ as follows:

$$\sum_{n=1}^{2^{k-1}M-1} \sum_{m=0}^{M-1} c_{nm} D_s^q \psi_{n,m}(x_i) = f(x_i) + \sum_{n=1}^{2^{k-1}M-1} \sum_{m=0}^{M-1} c_{nm} p(x_i) \psi_{n,m}(x_i) \quad (9)$$

$$+ \sum_{n=1}^{2^{k-1}M-1} \sum_{m=0}^{M-1} c_{nm} \int_0^{x_i} k(x_i, s) \psi_{n,m}(s) ds. \quad (10)$$

The best choice of the $x_i$ points are the zeros of the shifted Chebyshev polynomials of degree $2^{k-1}M - 1$ in the interval $[0,1]$ that is $x_i = \frac{s_i + \frac{1}{2}}{2}$, where $s_i = \cos \left( \frac{(2i-1)\pi}{2^{k-1}M-1} \right), i = 1, \ldots, 2^{k-1}M - 1.$

Approximating $D^q_s \psi_{n,m}$ using Diethelm method [3] on the representation that has been given by equation (2), we get

$$D^q_s \psi_{n,m}(x_i) = \int_0^{x_i} \frac{\psi_{n,m}(s) - \psi_{n,m}(0)}{(x_i-s)^{1+q}} ds = \frac{x_i^{-q}}{\Gamma(-q)} \int_0^1 \frac{\psi_{n,m}(x_i - ws) - \psi_{n,m}(0)}{w^{1+q}} dw$$

$$\approx \sum_{r=0}^L \alpha_r (\psi_{n,m}(x_i - x/rL) - \psi_{n,m}(0)).$$

where $L \in \mathbb{N}$ and the weights $\alpha_r$ are given by

$$q(1-q) L^{-q} \frac{\Gamma(-q)}{x_i^q} \alpha_r = \begin{cases} 
-1, & \text{if } r = 0, \\
2r^{1-q} - (r - 1)^{1-q} - (r + 1)^{1-q}, & \text{if } r = 1, 2, \ldots, L - 1, \\
(q - 1)r^{-q} - (r - 1)^{1-q} + r^{-1}, & \text{if } r = L.
\end{cases}$$
Then equation (8) becomes
\[
\sum_{n=1}^{2^{k-1}M-1} \sum_{m=0}^{L} \sum_{r=0}^{M-1} \alpha_r (\psi_{n,m}(x_i - x_i r/L) - \psi_{n,m}(0)) c_{nm} = f(x_i) + \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} p(x_i) \psi_{n,m}(x_i) \quad (11)
\]
+ \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \int_0^{x_i} k(x_i, s) \psi_{n,m}(s) ds \quad (12)

Combine equation (6) and (9) to obtain \(2^{k-1}M\) linear equations from which we can compute values for the unknown coefficients, \(c_{nm}\).

### 4 Numerical Examples

To demonstrate the effectiveness of the proposed method we consider here some fractional integro-differential equations. We use the software Maple to get the numerical results.

**Example 4.1.** Consider the following fractional integro-differential equation
\[
D_{0.75}^0 y(t) = \frac{t^{0.25}}{\Gamma(1.25)} - t^2 - t - \frac{1}{3} t^4 - \frac{1}{2} t^3 + ty(t) + \int_0^t tsy(s) ds, \quad (13)
\]
with the initial condition
\[
y(0) = 1, \quad (14)
\]
and the exact solution of (10) and (11) is \(y(t) = t + 1\).

We solved the linear system that was obtained by equations (6) and (9) with \(k = 2\) and \(M = 2\) to get
\[
c_{10} = 0.88388, c_{11} = 0.10206, c_{20} = 1.2374, c_{21} = 0.10206.
\]
Then the approximate solution will be
\[
\sum_{n=1}^{2} \sum_{m=0}^{1} c_{nm} \psi_{n,m}(t) = \begin{cases} 
0.88388(\sqrt{2}) + 0.10206(\sqrt{6})(4t - 1) = 0.99997t + 1.00005, & 0 \leq t < 0.5, \\
1.2374(\sqrt{2}) + 0.10206(\sqrt{6})(t - 3) = 0.99997t + 0.99996, & 0.5 \leq t \leq 1.
\end{cases}
\]
It is clear that the approximate solution almost coincides with the analytic solution.

**Example 4.2.** Consider the following fractional integro-differential equation
\[
D_{0.25}^0 y(t) = \frac{6t^{2.75}}{\Gamma(3.75)} - \frac{3}{10} t^2 e^t - \frac{1}{5} t^2 e^t y(t) + \int_0^t e^t sy(s) ds, \quad (15)
\]
with the initial condition

\[ y(0) = 1, \quad (16) \]

and the exact solution of (12) and (13) is \( y(t) = t^3 + 1 \).

We use the Legendre wavelets method with \( k = 1 \) and \( M = 6 \) to obtain

\[ c_{10} = 1.2507, \ c_{11} = 0.26054, \ c_{12} = 0.11211 \]

\[ c_{13} = 0.01895, \ c_{14} = 8.129 \times 10^{-10}, \ c_{15} = -1.8037 \times 10^{-10}. \]

Then the approximate solution will be

\[
\sum_{m=0}^{5} c_{1m} \psi_{n,m}(t) = -0.15075 \times 10^{-6} t^5 + 0.5476 \times 10^{-6} t^4
\]

\[
+ 1.00282 t^3 + 0.8307 \times 10^{-6} t^2 - 0.312 \times 10^{-7} t + 1.
\]

We see that the approximate solution is almost overlapping the exact solution.

Example 4.3. Consider the following fractional integro-differential equation

\[
D_0^ {0.5} y(t) = 2 e^{-t} \int_0^t \frac{e^{s}}{\sqrt{s}} ds - e^t \sin ty(t) + \int_0^t e^s \cos s y(s) ds, \quad (17)
\]

with the initial condition

\[ y(0) = 2, \quad (18) \]

and the exact solution of (14) and (15) is \( y(t) = 2e^{-t} \).

Solving the linear system that was obtained by equations (6) and (9) with \( k = 1 \) and \( M = 8 \), we get the approximate solution

\[
\sum_{m=0}^{7} c_{1m} \psi_{n,m}(t) = 2 - 2t + 1.0003t^2 - 0.3336t^3 + 0.8310 \times 10^{-1} t^4
\]

\[
- 0.1667 \times 10^{-1} t^5 + 0.2778 \times 10^{-2} t^6 - 0.3968 \times 10^{-3} t^7.
\]

This approximate solution shows excellent agreement with the Taylor polynomial of degree 7 of the function \( 2e^{-t} \).
5 Conclusion

In this work we illustrated a numerical algorithm for solving fractional integro-differential equations using Legendre wavelets method. We derived a system of equations that characterizing the numerical solution. It has been numerically demonstrated that the proposed method is effective and easy to use.

References


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