Total Coloring of
Certain Double Vertex Graphs

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Abstract
In this paper we investigate the total coloring number for double vertex graphs of some of the most common classes of graphs.

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1 Introduction

Total coloring was introduced by Behzad and Vizing in numerous occasions between 1964 and 1968 [4]. Behzad stated the total coloring number of several classes of graphs in his Ph.D. thesis called ”Graphs and their chromatic numbers”. In this paper we give total coloring number for double vertex graphs of some of the most common classes of graphs.

We consider (simple) graphs which are finite, undirected with no loops or multiple edges.

A total coloring of a graph G is a coloring of all elements of G, i.e. vertices and edges, so that no two adjacent or indecent elements receive the same color. The minimum number of colors is called the total chromatic number $\chi_T(G)$ of G [2].
The double vertex graph $U_2(G)$ is the graph whose vertex set consists of all 2-subsets of $V$ such that two distinct vertices $\{x, y\}$ and $\{u, v\}$ are adjacent if and only if $|\{x, y\} \cap \{u, v\}| = 1$ and if $x = u$, then $y$ and $v$ are adjacent in $G$ [5].

The notations which are used in the study are given below:

$\Delta(G)$: Maximum vertex degree of $G$.

$P_n$: Path of order $n$.

$C_n$: Cycle of order $n$.

$S_{1,n}$: Star of order $(n+1)$.

$K_{m,n}$: Complete bipartite graph of order $(m+n)$.

$N_n$: Null graph of order $n$. (which is a graph with no edges).

## 2 Total Coloring Of Double Vertex Graphs

In this chapter, we discuss some of the theorems about total coloring and we give total chromatic number for double vertex graphs of the most common classes of graphs.

**Theorem 2.1** [5] The double vertex graph of a graph is a cycle if and only if $G = K_3$ or $G = K_{1,3}$.

**Theorem 2.2** [5] If $G$ is a connected graph, then the double vertex graph of $G$ is a tree if and only if $G = K_3$ or $G = P_3$.

**Theorem 2.3** [5] If $G$ is a connected graph, then its double vertex graph is bipartite if and only if $G$ is bipartite.

**Theorem 2.4** The total chromatic number of double vertex graph of $G$ is,

$$\chi_T[U_2(G)] \geq \Delta[U_2(G)] + 1.$$ 

PROOF: For total coloring of the double vertex graph with the minimum color, the maximum degree vertex and the indecent edge(s) to this vertex must be colored. So the total chromatic number of graph $U_2(G)$ is at least one more than the maximum degree of $U_2(G)$.

**Theorem 2.5** There is a relation between total chromatic number of double vertex graph of $G$ and total chromatic number of double vertex of subgraph $H$ of $G$,

$$\chi_T[U_2(G)] \geq \chi_T[U_2(H)].$$

PROOF: The number of vertices and edges of the graph $H$ is less than the number of vertices and edges of the graph $G$. The maximum vertex degree of the double vertex graph $H$ is less than or equal to the maximum vertex degree of the double vertex graph $G$. So the number of edge coloring and
vertex coloring of graph $U_2(H)$ is less than or equal to graph $U_2(G)$. For the total coloring, both vertex coloring and edge coloring are needed so the total chromatic number of the graph $U_2(H)$ is at most total chromatic number of the graph $U_2(G)$.

**Theorem 2.6** [3] The total chromatic number of $P_n$ is,

$$\chi_T(P_n) = \Delta(P_n) + 1 = 3$$

**Theorem 2.7** [1] The total chromatic number of $C_n$ is,

$$\chi_T(C_n) = \begin{cases} 
3, & n \equiv 0 \pmod{3} \\
4, & \text{otherwise}
\end{cases}$$

**Theorem 2.8** [1] The total chromatic number of $K_{m,n}$ is,

$$\chi_T(K_{m,n}) = \begin{cases} 
\Delta(K_{m,n}) + 1, & m \neq n \\
\Delta(K_{m,n}) + 2, & m = n
\end{cases}$$

**Theorem 2.9** The total chromatic number of $U_2(P_n)$ is,

$$\chi_T[U_2(P_n)] = 5, (n > 4).$$

PROOF: For total coloring of the double vertex graph of paths with the minimum color, the maximum degree vertex and the indecent edge to this vertex must be colored. The maximum degree of graph $U_2(P_n)$ is 4 ($n > 4$). For edge coloring 4 colors are enough and one more color is needed to color a vertex, indecent to these edges. So (4 + 1 = 5), $\chi_T[U_2(P_n)] = 5$. And the maximum degree of graph $U_2(P_3)$ and graph $U_2(P_4)$ is (n-1). So (n-1+1 = n), $\chi_T[U_2(P_3)] = 3$ and $\chi_T[U_2(P_4)] = 4$.

**Theorem 2.10** The total chromatic number of $U_2(C_n)$ is,

$$\chi_T[U_2(C_n)] = 5, (n > 3).$$

PROOF: For total coloring of the double vertex graph of cycles with the minimum color, the maximum degree vertex and the indecent edge to this vertex must be colored. The maximum degree of graph $U_2(C_n)$ is 4 ($n > 3$). For edge coloring 4 colors are enough and one more color is needed to color a vertex, indecent to these edges. So (4 + 1 = 5), $\chi_T[U_2(C_n)] = 5$. And the maximum degree of graph $U_2(C_3)$ is 2 so (2 + 1 = 3), $\chi_T[U_2(C_3)] = 3$.

**Theorem 2.11** The total chromatic number of $U_2(S_{1,n})$ is,

$$\chi_T[U_2(S_{1,n})] = n, (n > 2).$$
PROOF: For total coloring of the double vertex graph of stars with the minimum color, the maximum degree vertex and the indecent edge to this vertex must be colored. For \( n > 2 \), the maximum degree of graph \( U_2(S_{1,n}) \) is \((n-1)\). For edge coloring, \((n-1)\) colors are enough and one more color is needed to color a vertex, indecent to these edges. So \((n-1+1 = n)\) colors are needed. For \( n \leq 2 \), \( U_2(S_{1,2}) = U_2(P_3) = P_3 \) so \( \chi_T(P_3) = \chi_T[U_2(S_{1,2})] = 3 \) (from Theorem 2.6). \( U_2(S_{1,1}) = N_1 \) so \( \chi_T(N_1) = \chi_T[U_2(S_{1,1})] = 1 \)

**Theorem 2.12** The total chromatic number of \( U_2(K_{m,n}) \) is,

\[ \chi_T[U_2(K_{m,n})] = \Delta(K_{m,n}) + 1. \]

PROOF: \( K_{m,n} \) is a complete bipartite graph and its double vertex graph is also bipartite graph from Theorem 2.3. Whenever \( m=n \) or \( m \neq n \), the double vertex graph of \( K_{m,n} \) is bipartite and \( m \) and \( n \) of \( U_2(K_{m,n}) \) are never equal. So from Theorem 2.8,

\[ \chi_T[U_2(K_{m,n})] = \Delta(K_{m,n}) + 1 \]

**Theorem 2.13**

\[ X_T[U_2(P_3)] = X_T(P_3) = 3 \]

PROOF: The double vertex graph of a path \( P_3 \) is a path \( P_3 \). So the total chromatic numbers of \( P_3 \) and \( U_2(P_3) \) graphs are equal.

**Theorem 2.14**

\[ X_T[U_2(K_3)] = X_T(C_3) = 3, \]
\[ X_T[U_2(K_{1,3})] = X_T(C_6) = 3. \]

PROOF: From Theorem 2.1 the double vertex graph of a complete graph \( K_3 \) is a cycle \( C_3 \). From Theorem 2.7 for \( n=3k \), \( X_T(C_n) = 3 \) so \( X_T[U_2(K_3)] = X_T(C_3) = 3 \). From Theorem 2.1 the double vertex graph of a complete bipartite graph \( K_{1,3} \) is a cycle \( C_6 \). From Theorem 2.7 for \( n=3k \), \( X_T(C_n) = 3 \) so \( X_T[U_2(K_3)] = X_T(C_3) = 3 \).

### 3 Conclusions

In this study, total coloring and total chromatic number for double vertex graphs of some of the most common classes of graphs are discussed. It is not easy to evaluate the total chromatic number of graphs. The most common classes of graphs’ total chromatic number are used to evaluate the total chromatic number of graphs. Through this idea we used some of the total chromatic number of certain graphs to evaluate the total chromatic number of double vertex graphs of the some certain graphs.
References


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