Exact Solutions for the Ito Equation

by the sn-ns Method

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Abstract

In this paper we derive some exact solutions for the Ito equation. These solutions are obtained by the sn-ns method, which is a generalization of the tanh-coth method.

Keywords: sn-ns method, Ito equation, soliton, traveling wave solution, nonlinear pde, fifth-order KdV

1 Introduction

A large variety of physical, chemical, and biological phenomena is governed by nonlinear evolution equations. The analytical study of nonlinear partial differential equations was of great interest during the last decades. Investigations of traveling wave solutions of nonlinear equations play an important role
in the study of nonlinear physical phenomena. The importance of obtaining the exact solutions, if available, of those nonlinear equations facilitates the verification of numerical solvers and aids in the stability analysis of solutions. In this paper we show solutions for Ito’s equation by using the sn-ns method [1], which is a generalization of the well known tanh-coth method.

The general fifth-order KdV equation reads

$$u_t + \omega u_{xxxx} + \alpha u^2 u_x + \beta u_x u_{xx} + \gamma u u_{xxx} = 0,$$

(1)

where $\alpha$, $\beta$, $\gamma$ and $\omega$ are arbitrary real parameters. Ito’s equation results from (1) when $\alpha = 2$, $\beta = 6$, $\gamma = 3$ and $\omega = 1$. Thus, the Ito equation reads [2]

$$u_t + u_{xxxx} + 2u^2 u_x + 6u_x u_{xx} + 3 u u_{xxx} = 0,$$

(2)

To solve the equation (2) we first unite the independent variables $x$ and $t$ into one wave variable $\xi = k(x + \lambda t + \xi_0)$ to carry the PDE (2) into the ODE (ordinary differential equation)

$$u'(\xi)(\lambda + 2u^2(\xi) + 6k^2u''(\xi)) + 3k^2u(\xi)u'''(\xi) + k^4u'''''(\xi) = 0.$$

(3)

so that $u(x,t) = v(\xi) = v(x + \lambda t)$. Integrating equation (3) with respect to $\xi$ gives following fourth order nonlinear ode

$$4u^3(\xi) + 9k^2[u'(\xi)]^2 + 6u(\xi)(\lambda + 3k^2u''(\xi)) + 6k^4u'''''(\xi) = C,$$

(4)

where $C$ is the constant of integration.

2 Solutions by the sn-ns method

In view of the sn-ns method [1], we seek solutions of equation (3) in the form

$$v(\xi) = a_0 + a_1 \text{sn}(\xi, m) + b_1 \text{ns}(\xi, m) + a_2 \text{sn}^2(\xi, m) + b_2 \text{ns}^2(\xi, m),$$

(5)
Substituting (5) into the ODE (4) results in a polynomial equation in powers of \( \varphi = \text{sn}(\xi,m) \). We collect all coefficients of powers of \( \varphi \) in the resulting equation where these coefficients have to vanish. This will give a system of algebraic equations involving the parameters \( a_i, b_i, \lambda \) and \( k \). Having determined these parameters. After substitution we obtain a large algebraic system. Solving it with the aid of a computer we obtain following solutions to Ito equation (2):

After solving this system, we get the following solutions:

1. \( a_0 = 0, a_1 = 0, a_2 = 30k^2, b_1 = 0, b_2 = -30k^2, C = 0, \lambda = 1152k^4 \), \( m = \sqrt{-1} \):

\[
u_1(x, t) = 30k^2 \left( \text{sn}^2(\xi, \sqrt{-1}) - \text{ns}^2(\xi, \sqrt{-1}) \right), \quad \xi = k \left( x + 1152k^4t \right).
\]

(6)

2. \( a_0 = 10k^2 (m^2 + 1), a_1 = 0, a_2 = -30k^2m^2, b_1 = 0, b_2 = -30k^2, C = 1760k^6 \left( m^6 - 33m^4 - 33m^2 + 1 \right), \lambda = -96k^4 \left( m^4 + 14m^2 + 1 \right) \):

\[
u_2(x, t) = 10k^2 \left( m^2 + 1 - 3m^2 \text{sn}^2(\xi, m) - 3m^2 \text{sn}^2(\xi, m) \right), \quad \xi = k \left( x - 96k^4 \left( m^4 + 14m^2 + 1 \right) t \right).
\]

(7)

3. \( a_0 = 10k^2 (m^2 + 1), a_1 = 0, a_2 = 0, b_1 = 0, b_2 = -30k^2, C = 880k^6 \left( 2m^6 - 3m^4 - 3m^2 + 2 \right), \lambda = -96k^4 \left( m^4 - m^2 + 1 \right) \):

\[
u_3(x, t) = 10k^2 \left( m^2 + 1 - 3m^2 \text{sn}^2(\xi, m) \right), \quad \xi = k \left( x - 96k^4 \left( m^4 - m^2 + 1 \right) t \right).
\]

(8)

4. \( a_0 = 10k^2 (m^2 + 1), a_1 = 0, a_2 = -30k^2m^2, b_1 = 0, b_2 = 0, C = 880k^6 \left( 2m^6 - 3m^4 - 3m^2 + 2 \right), \lambda = -96k^4 \left( m^4 - m^2 + 1 \right) \):

\[
u_4(x, t) = 10k^2 \left( m^2 + 1 - 3m^2 \text{sn}^2(\xi, m) \right), \quad \xi = k \left( x - 96k^4 \left( m^4 - m^2 + 1 \right) t \right).
\]
In particular, taking $m = 1$, we obtain hyperbolic solutions:

$$u_5(x, t) = 10k^2 \left( 2 - 3 \tanh^2 \xi - 3 \coth^2 \xi \right), \quad \xi = k(x - 1536k^4 t). \quad (9)$$

$$u_6(x, t) = 10k^2 \left( 2 - 3 \coth^2 \xi \right), \quad \xi = k(x - 96k^4 t). \quad (10)$$

$$u_7(x, t) = 10k^2 \left( 2 - 3 \tanh^2 \xi \right), \quad \xi = k(x - 96k^4 t). \quad (11)$$

On the other hand, choosing $m = 0$ gives

$$u_8(x, t) = 10k^2 \left( 1 - 3 \csc^2 \xi \right), \quad \xi = k(x - 96k^4 t). \quad (12)$$

Other exact solutions to Ito equation may be found in [3].

3 Conclusions

In this paper, by using the sn-ns method and the help of symbolic computation, we successfully obtained some exact solutions for the Ito equation (2). We think that the obtained solutions are new.

References


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